GENERALIZED Q-DEFORMED GELFAND-DICKEY STRUCTURES ON THE GROUP OF Q-PSEUDODIFFERENCE OPERATORS

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Abstract. We define the q-deformed Gelfand-Dickey bracket on the space of q-pseudodifference symbols which agrees with the Poisson Virasoro algebra of E.Frenkel and N.Reshetikhin and its generalizations and prove its uniqueness (in a natural class of quadratic Poisson structures). The associated hierarchies of nonlinear q-difference equations are also constructed.

1. Introduction

It is well known that the generalized KdV hierarchy of non-linear differential equations admits several different realizations. The first one is associated with the algebra of pseudodifferential operators on the line (or on the circle). The famous construction assigns to each nonlinear evolution equation in this hierarchy a pair (L, A) of differential operators such that the evolution equation is equivalent to the Lax equation

$$\frac{dL}{dt} = [A, L]\,. \tag{1.1}$$

The space of differential operators admits several remarkable Poisson structures, and Lax equations are Hamiltonian with respect to each of them. The simplest one is the so called first Gelfand-Dickey bracket, which is a *linear* Poisson bracket naturally arising from the identification of the space of differential operators with the dual of the Lie algebra of integral operators [1, 12]. The next one is the celebrated second Gelfand-Dickey bracket (or, Adler-Gelfand-Dickey bracket) [1, 8]. This bracket is *quadratic*, and its geometric comprehension has required much work; it admits at least three different realization, and isomorphisms between them usually represent deep theorems. The first one, which appears naturally in the study of Lax equations (1.1), is based on the study of the Lie group of integral operators (more precisely, of its central extension [10]). This group comes equipped with the natural Sklyanin bracket which endows it with the structure of a Poisson-Lie group, and the second Gelfand-Dickey bracket is identified with the Sklyanin bracket on its Poisson subvariety. The second realization, which is

totally different, is based on the study of the center of the universal enveloping algebra $U(\widehat{\mathfrak{sl}(n)})$ of the central extension of the loop algebra of $\mathfrak{sl}(n)$ at the critical value of the central charge [3]. The third realization, finally, is naturally related to the alternative description of the generalized KdV hierarchy which is provided by the Drinfeld-Sokolov theory [2]. The two latter approaches provide a natural generalization of the second Gelfand-Dickey bracket for arbitrary semisimple Lie algebras; the corresponding Poisson algebras are then called classical W-algebras.

Nonlinear differential equations (1.1) admit natural difference or q-difference analogues; their Hamiltonian treatment is more or less parallel to the differential case, although there arise some new and unexpected phenomena. As it happens, all three different constructions of the classical W-algebras referred to above have their natural q-difference counterparts. Historically, the first one to arise was based on the study of the Poisson structure on the center of the quantized universal enveloping algebra $U_q(\widehat{\mathfrak{sl}(n)})$ [6]. The quantization parameter q is naturally identified with the modulus of the associated q-difference operator, $D_q f(z) = f(qz)$.

The same Poisson structure also arises as a result of the Drinfeld-Sokolov type reduction for the first order matrix q-difference equation [7, 17]. A nontrivial point in the reduction procedure is that it involves a new elliptic classical r-matrix (its introduction is prompted by the consistency conditions for the reduction); the modulus τ of the underlying elliptic curve is related to q via $q = \exp \pi i \tau$.

The goal of the present paper is to provide the q-difference counterpart of the last remaining construction which is based on the study of the algebra of q-pseudodifference symbols. We prove that for each $n \in \mathbb{N}$ there exists a unique quadratic Poisson structure of the natural r-matrix type on the space $\mathbb{M}_n$ of the n-th order q-difference operators with normalized highest term such that formal spectral invariants

$$H_m(L) = \frac{n}{m} \operatorname{Tr} L^{\frac{m}{n}}, \quad m \in \mathbb{N} \quad , (m,n) = 1, \tag{1.2}$$

of a difference operator $L = D^n + u_{n-1} D^{n-1} + \cdots + u_0$ are in involution and generate q-difference Lax equations

$$\frac{dL}{dt} = [A, L], \quad A = L^{\frac{m}{n}}_{(+)}; \tag{1.3}$$

moreover, this Poisson structure coincides with the one obtained via the q-difference Drinfeld-Sokolov reduction (or, equivalently, with the one obtained in [6] via the study of the center of $U_q(\widehat{\mathfrak{sl}(n)})$ at the critical level). The generalized q-deformed KdV hierarchy which corresponds to (1.3) was described earlier by E.Frenkel [5]; however, his approach to the description of the associated Poisson structure is different: he simply uses the Poisson bracket borrowed from [6] and does not discuss its construction via the r-matrix formalism for the algebra of q-difference operators. Let us also note that the lattice version of this Poisson

structure has been introduced (in a different context) by W.Oewel [14] who also considered the lattice analogues of the KdV and KP hierarchies. These lattice hierarchies are also studied in [15].

In the second part of this paper the Poisson structure on the space of q-difference operators is generalized to the case of q-pseudodifference operators of arbitrary complex degree; this construction is motivated by [10], [11]. We extend the algebra $\Psi\mathbf{D}_q$ of q-pseudodifference symbols by adjoining to it the outer derivation $ad \ln D$ and performing the associated central extension; the extended Lie algebra of q-integral operators gives rise to the Lie group

$$\widehat{G}_- = \bigcup_{\alpha\in\mathbb{C}} \widehat{G}_\alpha, \quad \widehat{G}_\alpha = \left\{ L|\ L = D^\alpha + \sum_{i=1}^{\infty} u_i D^{\alpha-i} \right\}. \tag{1.4}$$

If $\alpha \in \mathbb{C}$ is generic, i.e., satisfies $\frac{\alpha \ln q}{2\pi i} \notin \mathbb{Q}$, for all elements $L \in \widehat{G}_\alpha$ there exists a logarithm and hence we may define L^β for each $\beta \in \mathbb{C}$. In particular, $L^{\frac{m}{\alpha}} \in \widehat{G}_m$ for any $m \in \mathbb{N}$ and hence contains only integer powers of D; let $L^{\frac{m}{\alpha}}_{(+)}$ be its positive part. The equation

$$\frac{dL}{dt} = \left[L^{\frac{m}{\alpha}}_{(+)}, L \right] \tag{1.5}$$

preserves $\widehat{G}_\alpha$ and has an infinite family of conservation laws $H_n(L) = \frac{\alpha}{n} \mathrm{Tr}\, L^{\frac{n}{\alpha}}$, $n \in \mathbb{N}$. The flows (1.5) for different m commute each with other. We show that in a natural class of Poisson brackets on $\widehat{G}_\alpha$there exists a unique one with respect to which the equations (1.5) are induced by the Hamiltonians $H_m(L)$. For integer α this bracket may be restricted to $\mathbb{M}_\alpha$; this restriction coincides with the bracket constructed in the first part of the present paper. A similar class of equations has been considered in [11], but Poisson structures for them have not been proposed. We shall discuss the relation between these two construction below (see remark 3.1).

2. Nonlinear q-difference equations of the KdV type

2.1. **Notation.** Throughout the paper we shall use the following notation. Let $\hat{h}$ be the dilation operator,

$$\hat{h} f(z) = f(qz), \quad f \in \mathbb{C}\left(\left(z^{-1}\right)\right), \quad q \in \mathbb{C}, \quad |q| < 1. \tag{2.1}$$

We denote by $\Psi\mathbf{D}_q$ the algebra of q-pseudodifference operators; by definition, $\Psi\mathbf{D}_q$ consists of formal series of the form

$$A = \sum_{i=-\infty}^{N(A)} a_i\left(z\right) D^i, \quad a_i \in \mathbb{C}\left(\left(z^{-1}\right)\right) \tag{2.2}$$

with the commutation relation

$$D \cdot a = \left(\hat{h} a\right) \cdot D, \quad a \in \mathbb{C}\left(\left(z^{-1}\right)\right). \tag{2.3}$$

For $a \in \mathbb{C}\left(\left(z^{-1}\right)\right)$ we put

$$ {}^{h^l}a = \hat{h}^l a, \quad \forall l \in \mathbb{C}. \tag{2.4}$$

As a linear space, $\Psi\mathbf{D}_q$ is a direct sum of three subalgebras,

$$J_+ = \left\{ A \in \Psi\mathbf{D}_q | \; A = \sum_{i=1}^{N(A)} a_i\left(z\right) D^i, \quad a_i \in \mathbb{C}\left(\left(z^{-1}\right)\right) \right\}, \tag{2.5}$$

$$J_0 = \mathbb{C}\left(\left(z^{-1}\right)\right) \subset \Psi\mathbf{D}_q, \tag{2.6}$$

$$J_- = \left\{ A \in \Psi\mathbf{D}_q | \; A = \sum_{i=1}^{\infty} a_i\left(z\right) D^{-i}, \quad a_i \in \mathbb{C}\left(\left(z^{-1}\right)\right) \right\}. \tag{2.7}$$

Clearly, J_0 normalizes $J_\pm$ and hence $J_{(\pm)} = J_\pm + J_0$ is also a subalgebra. Let $P_\pm, P_0$ be the associated projection operators which project $\Psi\mathbf{D}_q$ onto $J_\pm, J_0$, respectively, parallel to the complement. Put $P_{(\pm)} = P_\pm + P_0$. For $A \in \Psi\mathbf{D}_q$ set $A_\pm = P_\pm A, \quad A_{(\pm)} = P_{(\pm)}A$, $\mathrm{Res} A = A_0 = P_0 A$. For $a \in \mathbb{C}\left(\left(z^{-1}\right)\right), \quad a = \sum_i a_i z^i$, we put

$$\int a(z)dz/z = a_0; \tag{2.8}$$

clearly, this formal integral is dilation invariant, i.e.,

$$\int a(z)dz/z = \int a(qz)dz/z. \tag{2.9}$$

For $A \in \Psi\mathbf{D}_q$ we define its formal trace by

$$Tr\ A = \int \mathrm{Res}\ Adz/z; \tag{2.10}$$

it is easy to see that $Tr\ AB = Tr\ BA$ for any $A, B \in \Psi\mathbf{D}_q$. We introduce an inner product in $\Psi\mathbf{D}_q$ by

$$\langle A, B\rangle = Tr\ AB, \quad A, B \in \Psi\mathbf{D}_q. \tag{2.11}$$

Clearly, this inner product is invariant and non-degenerate and the subalgebras $J_\pm$ are isotropic; moreover, it sets J_+ and J_- into duality, while $J_0 \simeq J_0^*$.

2.2. **Fractional powers of q-pseudodifference operators and Lax equations.** The fractional powers formalism which is described below is largely parallel to the standard pseudodifferential case. Let $\mathbb{M}_n \subset \Psi\mathbf{D}_q$ be the affine subspace consisting of q-difference operators of the form

$$L = D^n + u_{n-1}D^{n-1} + \cdots + u_0, \quad u_i \in \mathbb{C}\left(\left(z^{-1}\right)\right). \tag{2.12}$$

We are interested in Lax equations of the form

$$\frac{dL}{dt} = \left[A, L\right], \quad L \in \mathbb{M}_n. \tag{2.13}$$

For consistency, the commutator in the r.h.s must be a polynomial in D of degree $\leq n-1$. Let Z_L be the centralizer of L in $\Psi\mathbf{D}_q$. Put

$$\Omega_L = \{A \in \Psi\mathbf{D}_q | \deg [A, L] \leq n-1\}.$$

Proposition 2.1. *$M \in Z_L$ implies $M_{(+)} \in \Omega_L$.*

Proposition 2.2. *For any $L \in \mathbb{M}_n$ there exists a unique $P \in \Psi\mathbf{D}_q$ of the form $P = D + \sum_{i=0}^{\infty} p_i D^{-i}$ such that $P^n = L$.*

We set $P = L^{1/n}$.

Proposition 2.3. *Any element $M \in Z_L$ is uniquely represented as*

$$M = \sum_{i=-\infty}^{m(M)} c_i L^{\frac{i}{n}}, \quad c_i \in \mathbb{C}.$$

Propositions 2.1, 2.2 imply that Lax equations

$$\frac{dL}{dt} = [A, L], \quad L \in \mathbb{M}_n, \quad A = M_{(+)}, \quad M = \sum_{i=-\infty}^{m(M)} c_i L^{\frac{i}{n}}, \quad c_i \in \mathbb{C}, \tag{2.14}$$

are self-consistent; without loss of generality we may assume that $c_i = 0$ if $l|i$.

Lemma 2.4. *Equation (2.14) implies that*

$$\frac{d}{dt} L^{\frac{r}{n}} = \left[A, L^{\frac{r}{n}}\right] \text{ for any } r \in \mathbb{N}.$$

Lemma 2.4 immediately implies

Proposition 2.5. *Functionals*

$$H_m = \frac{n}{m} \mathrm{Tr}\, L^{\frac{m}{n}}, \quad m \in \mathbb{N},$$

are conservation laws for (2.14).

Proposition 2.6. *Let*

$$\begin{aligned} \frac{dL}{dt} &= \left[M_{(+)}, L\right], \quad M = \sum_{i=-\infty}^{m(M)} c_i L^{\frac{i}{n}}, \quad c_i \in \mathbb{C}, \\ \frac{dL}{d\tau} &= \left[\tilde{M}_{(+)}, L\right], \quad \tilde{M} = \sum_{i=-\infty}^{m(\tilde{M})} \tilde{c}_i L^{\frac{i}{n}}, \quad \tilde{c}_i \in \mathbb{C}, \end{aligned}$$

be two Lax equations associated with any two elements in Z_L. Then

$$\frac{d^2 L}{dt d\tau} = \frac{d^2 L}{d\tau dt};$$

in other words, (formal) flows generated by M, $\tilde{M}$ commute with each other.

2.3. **q-difference Lax equations as Hamiltonian systems.** In this section we shall describe a family $\{,\}_n$, $n \in \mathbb{N}$, of Poisson structures on $\Psi\mathbf{D}_q$; the bracket $\{,\}_n$ may be restricted to $\mathbb{M}_n \subset \Psi\mathbf{D}_q$ and Lax equations (1.1) are Hamiltonian with respect to this bracket. We shall see later that $\{,\}_n$ coincides with the q-deformed Gelfand-Dickey bracket [6, 17] associated with the Lie algebra $\mathfrak{sl}(n)$.

An accurate definition of the Poisson structure should begin with the description of a class of admissible functionals and of their derivatives. In the present context the algebra of observables $\mathcal{A}$ is generated by 'elementary' functionals which assign to a pseudodifference operator A the formal integrals of its coefficients,

$$\zeta_i^j(A) = \mathrm{Tr}\,\left(z^{-j} A D^{-i}\right).$$

By definition, a functional $\varphi \in \mathcal{A}$ is smooth if for each $L \in \mathbb{M}_n \subset \Psi\mathbf{D}_q$ there exists an element $X \in \Psi\mathbf{D}_q$ (called its linear gradient) such that

$$\langle d\varphi(L), X\rangle = \left(\frac{d}{dt}\right)_{t=0} \varphi(L + tX).$$

In applications, various functionals may be defined only on an affine subspace of $\Psi\mathbf{D}_q$; in that case the choice of the gradient (when it exists) is not unique (however, a canonical choice is frequently possible). It is easy to see that 'elementary' functionals are smooth; in a similar way, traces of fractional powers of a pseudodifference operator are smooth functionals defined on affine subspaces $\mathbb{M}_n$; the gradient of such a functional may be so chosen that

$$[d\varphi(L), L] = 0.$$

Along with the linear gradient of a functional we shall frequently use its left and right gradients ∇, ∇' which are formally defined by

$$\begin{aligned}\langle \nabla\varphi(L), X\rangle &= \left(\tfrac{d}{dt}\right)_{t=0} \varphi\left((1 + tX)\, L\right),\\ \langle \nabla'\varphi(L), X\rangle &= \left(\tfrac{d}{dt}\right)_{t=0} \varphi\left(L\,(1 + tX)\right);\end{aligned}$$

obviously, $\nabla\varphi(L) = L d\varphi(L)$, $\quad \nabla'\varphi(L) = d\varphi(L) L$. A functional $\varphi \in \mathcal{A}$ is called invariant if $\nabla\varphi = \nabla'\varphi$.

Let us put $\mathfrak{d} = \Psi\mathbf{D}_q \oplus \Psi\mathbf{D}_q$ (direct sum of two copies); we introduce an invariant inner product in $\mathfrak{d}$ by

$$\left\langle\!\!\left\langle \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \right\rangle\!\!\right\rangle = \langle X_1, Y_1\rangle - \langle X_2, Y_2\rangle. \tag{2.15}$$

For a functional φ let us write $D\varphi = (\nabla\varphi, \nabla'\varphi) \in \mathfrak{d}$. We shall consider a class of Poisson brackets on $\Psi\mathbf{D}_q$ which depend bilinearly on left and right gradients of their arguments. In a very general way, such a bracket may be written as

$$\{\varphi, \psi\} = \langle\langle R D\varphi, D\psi\rangle\rangle,$$

where $R \in End\mathfrak{d}$, $R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.[1]

We shall postpone the discussion of the Jacobi identity for this class of brackets until part 3. Note only that it holds for all brackets constructed below.

A natural additional condition on this class of brackets is the involutivity of invariant functionals. It is easy to see that this condition (which allows to use formal traces to generate commuting Hamiltonian flows) is equivalent to the following simple constraint:

$$A + B = C + D.$$

A similar class of Poisson brackets is also defined in the pseudodifferential case. In this latter case, there is a simple standard choice of the operators A, B, C, D: $A = D$, $B = C = 0$; moreover, the operators $A = D$ should be skew symmetric and satisfy the modified classical Yang-Baxter equation. The standard choice is $A = \frac{1}{2}\left(P_{(+)} - P_{-}\right)$ (it corresponds to the second Gelfand-Dickey bracket, which is a special case of the general Sklyanin bracket). In the q-pseudodifference case this simple choice is no longer possible; indeed, the standard classical r-matrix

$$r_s = \frac{1}{2}\left(P_{(+)} - P_{-}\right)$$

is no longer skew, because of the different properties of the invariant inner product. Since the symmetric part of r_s is the projection operator onto the subspace of operators of order zero, it is natural to look for modified brackets of the form

$$\{\varphi, \psi\} = \left\langle\left\langle \begin{pmatrix} r + aP_0 & bP_0 \\ cP_0 & r + dP_0 \end{pmatrix} D\varphi, D\psi \right\rangle\right\rangle, \tag{2.16}$$

where $r = \frac{1}{2}\left(P_{+} - P_{-}\right)$ and a, b, c, d are linear operators acting in J_0 which satisfy

$$a = -a^*, \quad d = -d^*, \quad c^* = b.$$

In other words, the bracket (2.16) differs from the naive Gelfand-Dickey bracket by a 'perturbation term' which is acting only on the constant terms of the gradients (cf. [16]). We shall see below that for any choice of a, b, c, d this bracket satisfies the Jacobi identity. The additional conditions which allow to fix the choice of the bracket completely are given by the following uniqueness theorem.

Theorem 2.7. *There exists a unique Poisson bracket of the form (2.16) on $\Psi\mathbf{D}_q$ such that*

1) *the affine subspace $\mathbb{M}_n$ is a Poisson submanifold;*
2) *the Hamiltonians $H_m = \frac{n}{m}\mathrm{Tr}\ L^{\frac{m}{n}}$, $m \in \mathbb{N}$, are in involution and give rise to Lax equations*

$$\frac{dL}{dt} = \left[L_{(+)}^{\frac{m}{n}}, L\right], \quad L \in \mathbb{M}_n. \tag{2.17}$$

[1]Poisson brackets of this type were discussed by L.Freidel and J.-M.Maillet [4] and by L.Li and S.Parmentier [13].

This bracket is given by

$$\{\varphi,\psi\} = \left\langle\left\langle \begin{pmatrix} r + \frac{1}{2}\frac{1+\hat{h}^n}{1-\hat{h}^n}P_0' & -\frac{\hat{h}^n}{1-\hat{h}^n}P_0' + \frac{1}{2}P_{00} \\ \frac{1}{1-\hat{h}^n}P_0' + \frac{1}{2}P_{00} & r - \frac{1}{2}\frac{1+\hat{h}^n}{1-\hat{h}^n}P_0' \end{pmatrix} D\varphi, D\psi \right\rangle\right\rangle . \tag{2.18}$$

Remark 2.1. *Although the Poisson bracket satisfying the conditions of the theorem is unique, there remains some freedom in the choice of the corresponding r-matrix. The reason is that the gradients $D\varphi, D\psi$ are not arbitrary, namely, they belong to a family of isotropic linear subspaces in $\mathfrak{d}$; hence R is defined only up to an operator whose bilinear form identically vanishes on all such subspaces. As an example note that the bracket (2.18) may be also written in the form*

$$\{\varphi,\psi\} = \left\langle\left\langle \begin{pmatrix} P_+ + \frac{1}{1-\hat{h}^n}P_0' & -\frac{\hat{h}^n}{1-\hat{h}^n}P_0' \\ \frac{1}{1-\hat{h}^n}P_0' & P_+ - \frac{\hat{h}^n}{1-\hat{h}^n}P_0' \end{pmatrix} D\varphi, D\psi \right\rangle\right\rangle . \tag{2.19}$$

Remark 2.2. *Just as in the pseudodifferential case we may linearize the quadratic bracket (2.18) at the unit element of $\Psi\mathbf{D}_q$; the resulting bracket $\{\cdot,\cdot\}_1$ is linear; it is given by*

$$\{\varphi,\psi\}_1(X) = -\langle [r_s d\varphi, d\psi] + [d\varphi, r_s d\psi], X\rangle , \tag{2.20}$$

i.e., it is the Lie-Poisson bracket associated with the r-matrix r_s. The brackets (2.18) and (2.20) are compatible, i.e., their linear combinations are also Poisson brackets. Thus we have a 1-parameter family of quadratic Poisson brackets:

$$\{\varphi,\psi\}_\alpha = \{\varphi,\psi\} + \alpha\{\varphi,\psi\}_1 . \tag{2.21}$$

As usual, dynamical systems generated by the Hamiltonians H_m are bihamiltonian*; namely, the vector field generated by H_m with respect to the quadratic bracket (2.18) coincides with the vector field generated by H_{m+n} with respect to the linear bracket (2.20). Functionals H_m, $m \le n$, are Casimir functions for the bracket (2.20).*

Proof of the theorem. The gradients of H_m are given by

$$\nabla H_m = \nabla' H_m = L^{\frac{m}{n}},$$

hence the Hamiltonian equation generated by H_m with respect to the bracket (2.16) is given by

$$\frac{dL}{dt} = \left([r + (a+b)P_0]\ L^{m/n}\right)\cdot L - L\cdot\left([r+(c+d)P_0]\ L^{m/n}\right);$$

since $\left[L, L^{\frac{m}{n}}\right] = 0$, we get

$$\begin{aligned}\frac{dL}{dt} &= \left(\left[P_{(+)} + \left(a+b-\tfrac{1}{2}\right)P_0\right]\ L^{m/n}\right)\cdot L \\ &\qquad -L\cdot\left(\left[P_{(+)} + \left(c+d-\tfrac{1}{2}\right)P_0\right]\ L^{m/n}\right).\end{aligned}$$

This equation reduces to the Lax form (2.17) if and only if the coefficients a, b, c, d are such that for any $L \in \mathbb{M}_n$

$$\left(\left[a+b-\frac{1}{2}\right]\left(L^{m/n}\right)_0\right)\cdot L = L\cdot\left(\left[c+d-\frac{1}{2}\right]\left(L^{m/n}\right)_0\right). \tag{2.22}$$

Lemma 2.8. *Condition (2.22) implies that* $a+b-1/2 = (c+d-1/2) = F$, *where* F *is a linear operator in* J_0 *with* $ImF \subseteq \mathbb{C}\cdot 1 \subset J_0$.

Lemma 2.8 together with the antisymmetry condition imply that

$$b = \frac{1}{2} - a + F, \quad c = \frac{1}{2} + a + F^*, \quad d = -a + F - F^*. \tag{2.23}$$

It is easy to see that the bilinear form of the operators F, F^* vanishes on the gradients $D\varphi = (\nabla\varphi, \nabla'\varphi)$ of arbitrary functionals and hence does not contribute to the Poisson bracket; indeed,

$$\begin{aligned}\left[\begin{array}{l}\text{the contribution from}\\ F, F^* \text{ to } \quad \{\varphi,\psi\}\end{array}\right] &= \left\langle\!\!\left\langle \begin{pmatrix} 0 & FP_0 \\ F^*P_0 & (F-F^*)\,P_0\end{pmatrix} D\varphi, D\psi \right\rangle\!\!\right\rangle \\ &= FP_0\,(\nabla'\varphi)\cdot(\mathrm{Tr}\nabla\psi - \mathrm{Tr}\nabla'\psi) + \\ &\quad + FP_0\,(\nabla'\psi)\cdot(\mathrm{Tr}\nabla\varphi - \mathrm{Tr}\nabla'\varphi) \\ &= 0, \text{ due to invariance of Tr.}\end{aligned}$$

Thus we get

$$\{\varphi,\psi\} = \left\langle\!\!\left\langle \begin{pmatrix} P_+ + \left(\frac{1}{2}+a\right)P_0 & \left(\frac{1}{2}-a\right)P_0 \\ \left(\frac{1}{2}+a\right)P_0 & P_+ + \left(\frac{1}{2}-a\right)P_0 \end{pmatrix} D\varphi, D\psi \right\rangle\!\!\right\rangle. \tag{2.24}$$

The condition that the affine subspace $\mathbb{M}_n$ is a Poisson subvariety allows to fix the remaining free operator a. This condition means that the functionals

$$\varphi_f(L) = \int \frac{dz}{z} u_n(z)\, f(z) \equiv \mathrm{Tr}\left(LD^{-n}f\right), \quad \forall f \in \mathbb{C}((z^{-1})) \tag{2.25}$$

are Casimir functions on $\mathbb{M}_n$, i.e.,

$$\{\varphi_f, \psi\}\,|_{\mathbb{M}_n} = 0 \tag{2.26}$$

for any $\psi \in \mathcal{A}$. From (2.24) we get

$$\{\varphi_f,\psi\} = \left\langle \left[\frac{1}{2}+a\right](\nabla\varphi_f)_0 + \left[\frac{1}{2}-a\right](\nabla'\varphi_f)_0\,, (\nabla\psi)_0 - (\nabla'\psi)_0 \right\rangle. \tag{2.27}$$

Note that for any $L \in \mathbb{M}_n$

$$(\nabla\varphi_f)_0 = f, \quad (\nabla'\varphi_f)_0 = \hat{h}^{-n}(f),$$

in other words, the constant terms of the gradients do not depend on L. Hence the condition (2.26) is reduced to the following one:

- For any $f \in \mathbb{C}((z^{-1}))$ and any $L \in \mathbb{M}_n$

$$\left\langle \left[\frac{1}{2}+a\right] f + \left[\frac{1}{2}-a\right] \ ^{h^{-n}}f, (\nabla\psi)_0 - (\nabla'\psi)_0 \right\rangle = 0. \tag{2.28}$$

The latter condition is reduced to

$$\left[\frac{1}{2}\left(1+\hat{h}^{-n}\right) + a\left(1-\hat{h}^{-n}\right)\right] f \in \mathbb{C} \quad \text{for all } f \in \mathbb{C}((z^{-1})). \tag{2.29}$$

Indeed, (2.29) results immediately from the following

Lemma 2.9. *For any $g \in \mathbb{C}((z^{-1}))$ such that $\int \frac{dz}{z} g(z) = 0$ there exists a functional $\psi_g \in \mathcal{A}$ such that for some $L \in \mathbb{M}_{\mathbf{n}}$*

$$\left(\nabla\psi\left(L\right)\right)_0 - \left(\nabla'\psi\left(L\right)\right)_0 = g. \qquad \triangle$$

The proof of this assertion is similar to that of lemma 3.12 below.
Conversely, (2.29) implies (2.28) due to invariance of the trace.

From (2.29) we obtain that $a = a_0 + \beta - \gamma^*$, where

$$a_0 = \frac{1}{2}\frac{1+h^n}{1-h^n}\left(1 - P_{00}\right)$$

and β, γ are one-dimensional linear operators in $\mathbb{C}((z^{-1}))$ with $Im\beta, \gamma \subset \mathbb{C}\cdot 1$. The antisymmetry of a implies $\beta = \gamma$. Thus we have

$$\begin{array}{ll} a = a_0 + \gamma - \gamma^*, & b = \frac{1}{2} - a_0 + F - \gamma + \gamma^*, \\ c = \frac{1}{2} + a_0 + F^* - \gamma^* + \gamma, & d = -a_0 + F - \gamma - F^* + \gamma^*. \end{array}$$

To conclude the proof let us observe that F, γ do not contribute to the Poisson bracket, which implies (2.18). More precisely, we have the following assertion:

Lemma 2.10. *Let f, g, h, k be linear operators in J_0 with images in the subspace of constants $\mathbb{C}\cdot 1 \subset J_0$. The r-matrices R and $R' = R + \Delta$ where*

$$\Delta = \begin{pmatrix} h - k^* & f + k^* \\ h + g^* & -g^* + f \end{pmatrix},$$

define the same Poisson bracket.

Now we shall prove that the bracket (2.18) coincides with the q-deformed Gelfand-Dickey bracket derived in [7, 17] via the q-deformed Drinfeld-Sokolov reduction procedure. Let us first of all briefly recall this reduction procedure.

Let us denote by $L\mathfrak{gl}(n)$ the loop algebra associated with $\mathfrak{gl}(n)$, i.e., the algebra of $n \times n$ matrices with coefficients in $\mathbb{C}((z^{-1}))$.

It is well known that a scalar q-difference equation of order n

$$L\psi_0 = 0, \quad L = D^n + u_{n-1}(z)D^{n-1} + \cdots + u_0(z),$$

is equivalent to a first order matrix equation

$$D\Psi = \mathcal{L}\Psi, \quad \Psi = \begin{pmatrix} \psi_0 \\ \vdots \\ \psi_{n-1} \end{pmatrix},$$

where the potential $\mathcal{L} \in L\mathfrak{gl}(n)$ has a special form. The standard choice for $\mathcal{L}$ is given by a companion matrix,

$$\mathcal{L} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -u_0 & -u_1 & \cdots & -u_{n-1} \end{pmatrix}. \tag{2.30}$$

This choice is not unique; a linear change of variables

$$\Psi \mapsto \Psi' = S\Psi,$$

where S a lower triangular matrix with coefficients in $\mathbb{C}((z^{-1}))$, induces a gauge transformation

$$\mathcal{L} \longmapsto \mathcal{L}' =^h S\mathcal{L}\,S^{-1}. \tag{2.31}$$

Let us denote by $\mathbb{Y}_n \subset L\mathfrak{gl}(n)$ the subvariety of all matrices of the form

$$\mathcal{L}' = \begin{pmatrix} * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & 1 \\ * & * & \cdots & * \end{pmatrix}. \tag{2.32}$$

It is easy to see that the gauge action (2.31) of the group $L\mathbf{N}_-(n)$ of lower triangular matrices with the coefficients in $\mathbb{C}((z^{-1}))$ preserves $\mathbb{Y}_n$.

Theorem 2.11. [7, 17]

1. *The gauge action of $L\mathbf{N}_-(n)$ on $\mathbb{Y}_n$ is free.*
2. *The set of companion matrices of the form (2.30) is a cross-section of this action.*

This theorem implies that the quotient $\mathbb{Y}_n/L\mathbf{N}_-(n)$ can be identified with $\mathbb{M}_n$. In [7, 17] a natural description of the quotient $\mathbb{Y}_n/L\mathbf{N}_-(n)$ in the framework of Poisson reduction has been proposed. Let us recall some basic notions.

Let $\mathcal{M}$ be a Poisson manifold. The action of a Lie group G on $\mathcal{M}$ is called *admissible* if the ring of G-invariant functions $I_G(\mathcal{M})$ is a Poisson subalgebra in $Fun(\mathcal{M})$. Assume that the quotient $\mathcal{M}/G$ is a smooth manifold, then $I_G(\mathcal{M}) \approx Fun(\mathcal{M}/G)$ and therefore the quotient $\mathcal{M}/G$ has a Poisson structure. The natural projection $\pi : \mathcal{M} \to \mathcal{M}/G$ is Poisson with respect to this bracket.

Proposition 2.12. *Let $V \subset \mathcal{M}$ be a submanifold preserved by the action of G. The quotient V/G is a Poisson submanifold in $\mathcal{M}/G$ if and only if the ideal $I_0 \subset I_G(\mathcal{M})$ of all G-invariant functions vanishing on V is a Poisson ideal in $I_G(\mathcal{M})$.*

In our setting $\mathcal{M} = L\mathfrak{gl}(n)$, $G = L\mathbf{N}_-(n)$, $V = \mathbb{Y}_n$. In order to define the q-deformed Drinfeld-Sokolov reduction we need to find a Poisson structure on $L\mathfrak{gl}(n)$ which satisfies the following conditions:

1. the gauge action of $L\mathbf{N}_-(n)$ on $L\mathfrak{gl}(n)$ is admissible;
2. the constraints defining the submanifold $\mathbb{Y}_n \subset L\mathfrak{gl}(n)$ generate a Poisson ideal in $I_{L\mathbf{N}_-(n)}(L\mathfrak{gl}(n))$.

The latter condition means that the Poisson brackets of the constraints with any function vanish on the constraints surface $\mathbb{Y}_n \subset L\mathfrak{gl}(n)$, i.e., the constraints are of the *first class,* according to Dirac.

As shown in [7, 17], these two conditions allow to fix the Poisson structure on $L\mathfrak{gl}(n)$ and the underlying classical r- matrix in an essentially unique way. To give an explicit formula for this bracket let us fix the following notation.

Let $L\mathfrak{n}_+(n), L\mathfrak{n}_-(n), L\mathfrak{h}(n)$ be the subalgebras of strictly upper triangular, strictly lower triangular and diagonal matrices in $L\mathfrak{gl}(n)$ respectively; let $\mathcal{P}_+, \mathcal{P}_-, \mathcal{P}_0$ be the corresponding projectors. Let R_s be the automorphism of $L\mathfrak{h}(n)$ given by

$$R_s\mathbf{diag}(\alpha_0, \ldots, \alpha_{n-1}) = \mathbf{diag}(\alpha_{n-1}, \alpha_0, \ldots, \alpha_{n-2});$$

(this is the automorphism of $L\mathfrak{h}$ induced by the *Coxeter element* of the Weyl group). Put $\theta = R_s\hat{h}$. The r-matrix

$$\hat{r} = \frac{1}{2}\left(\mathcal{P}_+ - \mathcal{P}_- + \frac{1+\theta}{1-\theta}\mathcal{P}_0'\right), \tag{2.33}$$

where

$$\mathcal{P}_0' = \mathcal{P}_0 - \frac{1}{n}\int \frac{dz}{z}\mathrm{Tr}, \tag{2.34}$$

satisfies modified classical Yang-Baxter equation and is skew symmetric with respect to the invariant inner product

$$\langle A(z), B(z)\rangle = \int \frac{dz}{z}\mathrm{Tr}A(z)B(z), \quad A(z), B(z) \in L\mathfrak{gl}(n). \tag{2.35}$$

The Poisson bracket on $L\mathfrak{gl}(n)$ which makes possible the q-deformed Drinfeld-Sokolov reduction is given by

$$\left\{\hat{\varphi}, \hat{\psi}\right\}_S = \left\langle\left\langle \begin{pmatrix} \hat{r} & -\hat{h}\hat{r}_+ \\ \hat{r}_-\hat{h}^{-1} & -\hat{r} \end{pmatrix} \begin{pmatrix} \nabla\hat{\varphi} \\ \nabla'\hat{\varphi} \end{pmatrix}, \begin{pmatrix} \nabla\hat{\psi} \\ \nabla'\hat{\psi} \end{pmatrix} \right\rangle\right\rangle \tag{2.36}$$

where $\hat{r}_\pm = \hat{r} \pm \frac{1}{2}id$.

On the reduced space $\mathbb{Y}_n/L\mathbf{N}_-(n)$ which we identify with $\mathbb{M}_n$ we obtain a Poisson bracket called the q-deformed (second) Gelfand-Dickey structure.

Theorem 2.13. *The bracket (2.18) coincides with the q-deformed Gelfand-Dickey structure* $\{\cdot,\cdot\}_q$.

Proof. For a functional f on $L\mathfrak{gl}(n)$ put

$$Z_f =^{h^{-1}} \nabla f - \nabla' f. \tag{2.37}$$

Let us denote by $L\mathfrak{b}_-(n)$ the subalgebra of lower triangular matrices in $L\mathfrak{gl}(n)$ with arbitrary diagonal elements.

Lemma 2.14. *A functional f is $L\mathbf{N}_-(n)$ - invariant if and only if $Z_f \in L\mathfrak{b}_-(n)$.*

Lemma 2.15. *The value of $\{f,g\}_S(\mathcal{L})$ at any $\mathcal{L}\in\mathbb{Y}_n$ does not depend on the $L\mathfrak{n}_-(n)$-components of df, dg provided that $Z_f, Z_g \in L\mathfrak{b}_-(n)$.*

For any $L\in\mathbb{M}_n$ we denote by $\mathcal{L}$ the corresponding companion matrix of the form (2.30). Let φ,ψ be any functionals on $\mathbb{M}_n$; by construction, the quotient Poisson structure on $\mathbb{Y}_n/L\mathbf{N}_-(n)\simeq\mathbb{M}_n$ is given by

$$\{\varphi,\psi\}_q(L) = \left\{\hat\varphi,\hat\psi\right\}_S(\mathcal{L}), \tag{2.38}$$

where $\hat\varphi,\hat\psi$ are any $L\mathbf{N}_-(n)$ - invariant functionals on $L\mathfrak{gl}(n)$ such that their restrictions on $\mathbb{Y}_n$ coincide with the pullbacks of φ and ψ, respectively:

$$\hat\varphi\mid_{\mathbb{Y}_n} = \pi^*\varphi, \quad \hat\psi\mid_{\mathbb{Y}_n} = \pi^*\psi. \tag{2.39}$$

To calculate the r.h.s. of (2.38) we need to know only the gradients $d\hat\varphi$, $d\hat\psi$ and not $\hat\varphi$, $\hat\psi$ themselves. The upper triangular components of the gradients are fixed by (2.39), and their strictly lower triangular components may be chosen arbitrarily, provided that $Z_{\hat\varphi}, Z_{\hat\psi}\in L\mathfrak{b}_-(n)$, in agreement with lemma 2.15.

Note that φ, ψ are defined only on $\mathbb{M}_n$, hence their gradients are defined modulo the annihilator $\widehat{\mathbb{M}}_{n-1}$of the tangent space to $\mathbb{M}_n$. To fix them we shall suppose that they have the form

$$d\varphi = \sum_{i=0}^{n-1} f_i D^{-i}, \quad f_i\in\mathbb{C}((z^{-1})). \tag{2.40}$$

Lemma 2.16. *The upper triangular component of $d\hat\varphi$ is given by*

$$d\hat\varphi_{pm}(\mathcal{L}) = -\mathrm{Res}\left(D^p d\varphi\left[LD^{-(m+1)}\right]_{(+)}\right), \quad m\geq p. \tag{2.41}$$

Let us define the strictly lower triangular components of $d\hat\varphi$ by the same formula. We need to verify that $Z_{\hat\varphi}\in L\mathfrak{b}_-(n)$; this results directly from the following lemma:

Lemma 2.17. *We have*

$$\begin{aligned}\nabla\hat{\varphi}_{pm}(\mathcal{L}) &= \delta_{n-1,p}\left(\nabla\varphi\left[LD^{-(m+1)}\right]_{(+)}\right)_0 - \\ &\qquad -\bar{\delta}_{n-1,p}\left(D^{p+1}d\varphi\left[LD^{-(m+1)}\right]_{(+)}\right)_0; \qquad (2.42)\end{aligned}$$

$$\left(Z_{\hat{\varphi}}\right)_{n-1,0}(\mathcal{L}) = {}^{h^{-1}}\left(\nabla\varphi\left[LD^{-1)}\right]_{(+)}\right)_0 - \left(D^{n-1}d\varphi\right)_0 u_0; \qquad (2.43)$$

$$\left(Z_{\hat{\varphi}}\right)_{n-1,m}(\mathcal{L}) = {}^{h^{-1}}\left(\nabla\varphi\left[LD^{-(m+1)}\right]_{(+)}\right)_0, \quad m \neq 0; \qquad (2.44)$$

$$\left(Z_{\hat{\varphi}}\right)_{p-1,0}(\mathcal{L}) = -\left(D^p\nabla'\varphi\right)_0, \qquad (2.45)$$

where $\bar{\delta} = 1-\delta$. *All other elements of* $Z_{\hat{\varphi}}$ *are zero.*

Taking into account that $Z_{\hat{\varphi}}, Z_{\hat{\psi}} \in L\mathfrak{b}_-(n)$, we get the following expression for the bracket (2.36)

$$\left\{\hat{\varphi},\hat{\psi}\right\}_S = \frac{1}{2}\left(\left\langle\frac{1+\theta}{1-\theta}Z^0_{\hat{\varphi}}, Z^0_{\hat{\psi}}\right\rangle + \left\langle{}^hZ_{\hat{\varphi}}, \nabla\hat{\psi}\right\rangle - \left\langle\nabla\hat{\varphi}, {}^hZ_{\hat{\psi}}\right\rangle\right), \qquad (2.46)$$

where $Z^0_{\hat{\varphi}} \equiv \mathcal{P}'_0 Z_{\hat{\varphi}}, \quad Z^0_{\hat{\psi}} \equiv \mathcal{P}'_0 Z_{\hat{\psi}}$.

Let us calculate the contribution of the first term in (2.46) to $\left\{\hat{\varphi},\hat{\psi}\right\}_S$.

Lemma 2.18. *The eigenfunctions of the operator* θ *are*

$$E_{m,\alpha} = z^m\mathbf{e}_\alpha, \quad m\in\mathbb{Z}, \quad \alpha = 0,\dots,n-1, \qquad (2.47)$$

where

$$\mathbf{e}_\alpha = \mathbf{diag}\left(1,\omega^{-\alpha},\dots,\omega^{-(n-1)\alpha}\right), \quad \omega = e^{\frac{2\pi i}{n}}. \qquad (2.48)$$

The corresponding eigenvalues $\xi_{m,\alpha}$ *are equal to*

$$\xi_{m,\alpha} = q^m\omega^\alpha. \qquad (2.49)$$

The eigenfunctions satisfy the condition

$$\langle E_{m,\alpha}, E_{l,\beta}\rangle = n\delta_{m,-l}\cdot\begin{cases}1, & \alpha = -\beta \bmod n,\\ 0, & \textit{in the other cases,}\end{cases} \qquad (2.50)$$

and form a basis in $L\mathfrak{h}(n)$.

We shall denote by $\sum'_{m,\alpha}$ the sum over all pairs $(m,\alpha)\neq(0,0)$, $m\in\mathbb{Z}$, $\alpha = 0,\dots,n-1$. Note that in the expansion of $Z^0_{\hat{\varphi}}, Z^0_{\hat{\psi}}$ with respect to the eigenbasis $E_{m,\alpha}$ the $E_{0,0}$-component is absent, hence

$$\left\langle\frac{1+\theta}{1-\theta}Z^0_{\hat{\varphi}}, Z^0_{\hat{\psi}}\right\rangle = \left\langle\sum_{m,\alpha}{}'\frac{1}{n}\frac{1+q^m\omega^\alpha}{1-q^m\omega^\alpha}E_{m,\alpha}\left\langle Z_{\hat{\varphi}}, E_{-m,n-\alpha}\right\rangle, Z_{\hat{\psi}}\right\rangle =$$

$$= \int \frac{dz}{z} \int \frac{dw}{w} \sum_{m,\alpha}{}' \frac{1}{n} \frac{1+q^m\omega^\alpha}{1-q^m\omega^\alpha} \left(\frac{z}{w}\right)^m \operatorname{Tr}\left(\mathcal{P}_0 Z_{\hat{\varphi}}(w) \cdot \mathbf{e}_{n-\alpha}\right) \operatorname{Tr}\left(\mathcal{P}_0 Z_{\hat{\psi}}(z) \cdot \mathbf{e}_\alpha\right). \tag{2.51}$$

Applying lemma 2.17 we obtain:

$$\begin{aligned} &\operatorname{Tr}\left(\mathcal{P}_0 Z_{\hat{\varphi}}(w) \cdot \mathbf{e}_{n-\alpha}\right) \operatorname{Tr}\left(\mathcal{P}_0 Z_{\hat{\psi}}(z) \cdot \mathbf{e}_\alpha\right) = \\ &= {}^{h^{-1}} P_0 \nabla \varphi(w) \cdot {}^{h^{-1}} P_0 \nabla \psi(z) + P_0 \nabla' \varphi(w) \cdot P_0 \nabla' \psi(z) - \\ &- \omega^{-\alpha} \cdot {}^{h^{-1}} P_0 \nabla \varphi(w) \cdot P_0 \nabla' \psi(z) - \omega^\alpha P_0 \nabla' \varphi(w) \cdot {}^{h^{-1}} P_0 \nabla \psi(z). \end{aligned} \tag{2.52}$$

We denote by $A_1^1, \dots, A_1^4$ the contributions of the corresponding terms of (2.52) to $\left\langle \frac{1+\theta}{1-\theta} Z^0_{\hat{\varphi}}, Z^0_{\hat{\psi}} \right\rangle$.

Lemma 2.19.

$$\frac{1}{n}\sum_{\alpha=0}^{n-1} \frac{1+q^m\omega^\alpha}{1-q^m\omega^\alpha} = \frac{1+q^{mn}}{1-q^{mn}}, \quad m \neq 0; \tag{2.53}$$

$$\frac{1}{n}\sum_{\alpha=0}^{n-1} \frac{1+\omega^\alpha}{1-\omega^\alpha} = 0 \tag{2.54}$$

$$\frac{1}{n}\sum_{\alpha=0}^{n-1} \frac{1+q^m\omega^\alpha}{1-q^m\omega^\alpha}\omega^\alpha = 2\frac{q^{m(n-1)}}{1-q^{mn}}, \quad m \neq 0; \tag{2.55}$$

$$\frac{1}{n}\sum_{\alpha=0}^{n-1} \frac{1+\omega^\alpha}{1-\omega^\alpha}\omega^\alpha = -\frac{n-2}{n} \tag{2.56}$$

$$\frac{1}{n}\sum_{\alpha=0}^{n-1} \frac{1+q^m\omega^\alpha}{1-q^m\omega^\alpha}\omega^{-\alpha} = 2\frac{q^m}{1-q^{mn}}, \quad m \neq 0; \tag{2.57}$$

$$\frac{1}{n}\sum_{\alpha=0}^{n-1} \frac{1+\omega^\alpha}{1-\omega^\alpha}\omega^{-\alpha} = \frac{n-2}{n} \tag{2.58}$$

Proof. Let us prove (2.53); formulae (2.55), (2.57) may be verified in the same way. We have

$$S_1 = \frac{1}{n}\sum_{\alpha=0}^{n-1} \frac{1+q^m\omega^\alpha}{1-q^m\omega^\alpha} = -1 + \frac{2}{n}\sum_{\alpha=0}^{n-1} \frac{1}{1-q^m\omega^\alpha} = -1 + \frac{2}{n}\sum_{i=0}^{\infty}\sum_{\alpha=0}^{n-1} q^{mi}\omega^{\alpha i},$$

but

$$\frac{1}{n}\sum_{\alpha=0}^{n-1} \omega^{\alpha i} = \begin{cases} 0, & i \neq jn, \quad j \in \mathbb{N}, \\ 1, & i = jn, \end{cases}$$

and hence

$$S_1 = -1 + 2\sum_{j=0}^{\infty} q^{mnj} = \frac{1+q^{mn}}{1-q^{mn}},$$

as desired.

To prove (2.54) note that

$$\frac{1+\omega^{n-\alpha}}{1-\omega^{n-\alpha}} = \frac{1+\omega^{-\alpha}}{1-\omega^{-\alpha}} = -\frac{1+\omega^{\alpha}}{1-\omega^{\alpha}},$$

therefore for odd n all terms in the sum (2.54) cancel each other completely. If n is even only the term $\dfrac{1+\omega^{n/2}}{1-\omega^{n/2}}$ survives, but evidently it is zero, since $\omega^{n/2} = -1$. Formulae (2.56), (2.58) immediately follow from (2.54).

Lemma 2.20.

$$A_1^1 = \left\langle \frac{1+\hat{h}^n}{1-\hat{h}^n} P_0' \nabla\varphi,\, \nabla\psi \right\rangle, \tag{2.59}$$

$$A_1^2 = \left\langle \frac{1+\hat{h}^n}{1-\hat{h}^n} P_0' \nabla'\varphi,\, \nabla'\psi \right\rangle, \tag{2.60}$$

$$A_1^3 = \left\langle \frac{-2}{1-\hat{h}^n} P_0' \nabla\varphi,\, \nabla'\psi \right\rangle - \frac{n-2}{n} \mathrm{Tr}\nabla\varphi \cdot \mathrm{Tr}\nabla'\psi, \tag{2.61}$$

$$A_1^4 = \left\langle \frac{-2\hat{h}^n}{1-\hat{h}^n} P_0' \nabla'\varphi,\, \nabla\psi \right\rangle + \frac{n-2}{n} \mathrm{Tr}\nabla'\varphi \cdot \mathrm{Tr}\nabla\psi. \tag{2.62}$$

Proof. We verify only (2.61), other formulae can be proved in the same way. We have:

$$\begin{aligned}
A_1^3 &= -\int \frac{dz}{z} \int \frac{dw}{w} \sum_{m,\alpha}{}' \frac{1}{n} \frac{1+q^m\omega^\alpha}{1-q^m\omega^\alpha} \omega^{-\alpha} \left(\frac{z}{w}\right)^m \cdot\ {}^{h^{-1}} P_0 \nabla\varphi(w) \cdot P_0 \nabla'\psi(z) \\
&= -\int \frac{dz}{z} \left[\int \frac{dw}{w} \sum_{\substack{m\in\mathbb{Z} \\ m\neq 0}} \frac{1}{n} \frac{2q^m}{1-q^{mn}} \omega^{-\alpha} \left(\frac{z}{w}\right)^m \cdot\ {}^{h^{-1}} P_0 \nabla\varphi(w) \right] \cdot P_0 \nabla'\psi(z) \\
&\quad -\int \frac{dz}{z} \left[\int \frac{dw}{w} \left(\frac{1}{n} \sum_{\alpha=0}^{n-1} \frac{1+\omega^\alpha}{1-\omega^\alpha} \omega^{-\alpha} \right) \cdot\ {}^{h^{-1}} P_0 \nabla\varphi(w) \right] \cdot P_0 \nabla'\psi(z) \\
&= -\int \frac{dz}{z} \left(\frac{2\hat{h}}{1-\hat{h}^n}\, {}^{h^{-1}} P_0' \nabla\varphi \right)(z) \cdot P_0 \nabla'\psi(z) - \\
&\quad -\frac{n-2}{n} \int \frac{dw}{w}\, {}^{h^{-1}} P_0' \nabla\varphi(w) \cdot \int \frac{dz}{z} P_0 \nabla'\psi(z) \qquad \text{(by lemma 2.19)} \\
&= \left\langle \frac{-2}{1-\hat{h}^n} P_0' \nabla\varphi,\, \nabla'\psi \right\rangle - \frac{n-2}{n} \mathrm{Tr}\nabla\varphi \cdot \mathrm{Tr}\nabla'\psi,
\end{aligned}$$

as desired.

Using lemma 2.20 and taking into account that $\mathrm{Tr}\nabla\varphi = \mathrm{Tr}\nabla'\varphi$, we obtain

$$\left\langle \frac{1}{2}\frac{1+\theta}{1-\theta}Z^0_{\hat{\varphi}}, Z^0_{\hat{\psi}} \right\rangle = \left\langle\left\langle \begin{pmatrix} \frac{1}{2}\frac{1+\hat{h}^n}{1-\hat{h}^n}P_0' & -\frac{\hat{h}^n}{1-\hat{h}^n}P_0' \\ \frac{1}{1-\hat{h}^n}P_0' & -\frac{1}{2}\frac{1+\hat{h}^n}{1+\hat{h}^n}P_0' \end{pmatrix} \begin{pmatrix} \nabla\varphi \\ \nabla'\varphi \end{pmatrix}, \begin{pmatrix} \nabla\psi \\ \nabla'\psi \end{pmatrix} \right\rangle\right\rangle. \tag{2.63}$$

It remains to calculate $\left\langle {}^h Z_{\hat{\varphi}}, \nabla\hat{\psi} \right\rangle - \left\langle \nabla\hat{\varphi}, {}^h Z_{\hat{\psi}} \right\rangle$.

Lemma 2.21.

$$\begin{aligned} \left\langle {}^h Z_{\hat{\varphi}}, \nabla\hat{\psi} \right\rangle &= \langle r\nabla\varphi, \nabla\psi\rangle - \langle r\nabla'\varphi, \nabla'\psi\rangle + \\ &+ \frac{1}{2}\mathrm{Tr}\left(P_0\nabla\varphi \cdot P_0\nabla\psi\right) - \frac{1}{2}\mathrm{Tr}\left(P_0\nabla'\varphi \cdot P_0\nabla'\psi\right). \end{aligned} \tag{2.64}$$

Proof. Taking into account that $\left[LD^{-n}\right]_{(+)} = 1$ and using lemma 2.17, we obtain

$$\left\langle {}^h Z_{\hat{\varphi}}, \nabla\hat{\psi} \right\rangle = A_2^1 + A_2^2 + A_2^3 + A_2^4, \tag{2.65}$$

where

$$\begin{aligned} A_2^1 &= \mathrm{Tr}\left(\sum_{p=0}^{n-2} \left(D^{p+1}\nabla'\varphi D^{-1}\right)_0 \left(Dd\psi\left[LD^{-(p+1)}\right]_{(+)}\right)_0 \right), \\ A_2^2 &= -\mathrm{Tr}\left(\sum_{p=0}^{n-2} \left(\nabla\varphi\left[LD^{-(m+1)}\right]_{(+)}\right)_0 \left(D^{m+1}d\psi\right)_0 \right), \\ A_2^3 &= \mathrm{Tr}\left({}^h\left[\left(D^{n-1}d\varphi\right)_0 u_0\right] \left(Dd\psi\right)_0 \right), \\ A_2^4 &= \mathrm{Tr}\left((\nabla\varphi)_0 (\nabla\psi)_0 \right). \end{aligned}$$

In transformations below we use the fact that $\mathrm{Tr}A_{(+)}B = \mathrm{Tr}AB_{(-)}$ and the following proposition:

Proposition 2.22. *Let $B \in \Psi\mathbf{D}_q$ be of the form*

$$B = \sum_{i=m}^{l} b_i D^{-i}, \quad b_i \in \mathbb{C}\left(\left(z^{-1}\right)\right),$$

then

$$\sum_{i=m}^{l} D^{-i}\left(D^i B\right)_0 = B.$$

Since $d\varphi$, $d\psi$ have the form (2.40), we obtain:

$$\begin{aligned}
A_2^1 &= \mathrm{Tr}\left(Dd\psi\left[L\sum_{p=0}^{n-2}D^{-(p+1)}\left(D^{p+1}\nabla'\varphi D^{-1}\right)_0\right]_{(+)}\right)\\
&= \mathrm{Tr}\left(Dd\psi\left[L\left(\nabla'\varphi D^{-1}\right)_- - LD^{-n}\left(D^n\nabla'\varphi D^{-1}\right)_0\right]_{(+)}\right)\\
&= \mathrm{Tr}\left(\left(Dd\psi\right)_{(-)}L\left(\nabla'\varphi D^{-1}\right)_-\right) - \mathrm{Tr}\left(Dd\psi\left(D^n\nabla'\varphi D^{-1}\right)_0\right)\\
&= \mathrm{Tr}\left(\left(Dd\psi - D\left(d\psi\right)_0\right)L\left(\nabla'\varphi D^{-1}\right)_-\right) - \mathrm{Tr}\left(\left(Dd\psi\right)_0\cdot\ {}^h\left(D^{n-1}\nabla'\varphi\right)_0\right)\\
&= \mathrm{Tr}\left(\left(\nabla'\psi - \left(d\psi\right)_0L\right)\nabla'\varphi_{(-)}\right) - \mathrm{Tr}\left(\left(Dd\psi\right)_0\cdot\ {}^h\left[\left(D^{n-1}d\varphi\right)_{(-)}L\right]\right)\\
&= \mathrm{Tr}\nabla'\varphi\nabla'\psi_{(+)} - \mathrm{Tr}\left(d\psi\right)_0L\nabla'\varphi - \mathrm{Tr}\left(\left(Dd\psi\right)_0\cdot\ {}^h\left[\left(D^{n-1}d\varphi\right)_{(0)}u_0\right]\right)\\
&= \mathrm{Tr}\nabla'\varphi\nabla'\psi_{(+)} - \mathrm{Tr}\nabla\varphi L\left(d\psi\right)_0 - A_2^3.
\end{aligned}\tag{2.66}$$

A_2^2 may be developed as follows:

$$\begin{aligned}
A_2^2 &= -\mathrm{Tr}\left(\sum_{p=0}^{n-2}\left(\nabla\varphi\left[LD^{-(m+1)}\right]_{(+)}\right)_0\left(D^{m+1}d\psi\right)_0\right)\\
&= -\mathrm{Tr}\left(\nabla\varphi\left[\sum_{p=0}^{n-2}LD^{-(m+1)}\left(D^{m+1}d\psi\right)_0\right]_{(+)}\right)\\
&= -\mathrm{Tr}\left(\nabla\varphi\left[L\left(d\psi - \left(d\psi\right)_0\right)\right]_{(+)}\right)\\
&= -\mathrm{Tr}\nabla\varphi\left(\nabla\psi\right)_{(+)} + \mathrm{Tr}\nabla\varphi L\left(d\psi\right)_0.
\end{aligned}\tag{2.67}$$

Substituting (2.66), (2.67) in (2.65) gives

$$\left\langle {}^hZ_{\hat\varphi}, \nabla\hat\psi\right\rangle = \mathrm{Tr}\nabla'\varphi\nabla'\psi_{(+)} - \mathrm{Tr}\nabla\varphi\left(\nabla\psi\right)_{(+)} + \mathrm{Tr}\left(\nabla\varphi\right)_0\left(\nabla\psi\right)_0$$

which immediately implies (2.64). Lemma 2.21 is proved.

For $\left\langle\nabla\hat\varphi, {}^hZ_{\hat\psi}\right\rangle$ we have a relation similar to (2.64), hence

$$\left\langle {}^hZ_{\hat\varphi}, \nabla\hat\psi\right\rangle - \left\langle\nabla\hat\varphi, {}^hZ_{\hat\psi}\right\rangle = 2\left(\langle r\nabla\varphi, \nabla\psi\rangle - \langle r\nabla'\varphi, \nabla'\psi\rangle\right).\tag{2.68}$$

Substituting (2.68) and (2.63) in (2.46) we obtain

$$\{\varphi,\psi\}_q = \left\langle\left\langle\begin{pmatrix} r + \frac{1}{2}\frac{1+\hat h^n}{1-\hat h^n}P_0' & -\frac{\hat h^n}{1-\hat h^n}P_0' \\ \frac{1}{1-\hat h^n}P_0' & r - \frac{1}{2}\frac{1+\hat h^n}{1+\hat h^n}P_0'\end{pmatrix}\begin{pmatrix}\nabla\varphi\\ \nabla'\varphi\end{pmatrix}, \begin{pmatrix}\nabla\psi\\ \nabla'\psi\end{pmatrix}\right\rangle\right\rangle.\tag{2.69}$$

This formula differs from (2.18) by the absence of $\pm\frac{1}{2}P_{00}$ in the non-diagonal elements of r-matrix, but the corresponding terms give no contribution to the

bracket because of the invariance of the inner product:

$$\left\langle \frac{1}{2}P_{00}\nabla'\varphi, \nabla\psi \right\rangle - \left\langle \frac{1}{2}P_{00}\nabla\varphi, \nabla'\psi \right\rangle = \frac{1}{2}\mathrm{Tr}\nabla'\varphi\mathrm{Tr}\nabla\psi - \mathrm{Tr}\nabla\varphi\mathrm{Tr}\nabla'\psi = 0.$$

Theorem 2.13 is proved.

Theorem 2.23. *The Poisson bracket (2.18) in terms of generating functions*

$$u_i(z) = \sum_{m=-\infty}^{N(u_i)} u_{im} z^m, \quad i = 0, \dots, n-1, \tag{2.70}$$

has the form

$$\begin{aligned} \{u_i(z), u_j(w)\} &= \sum_{\substack{m\in\mathbb{Z} \\ m\neq 0}} \frac{\left(1-q^{m(n-i)}\right)\left(1-q^{mj}\right)}{1-q^{mn}} \left(\frac{w}{z}\right)^m u_i(z)\, u_j(w) + \\ &+ \sum_{r=1}^{\min(n-i,j)} \delta\left(\frac{wq^r}{z}\right) u_{i+r}(w)\, u_{j-r}(z) - \\ &- \sum_{r=1}^{\min(n-i,j)} \delta\left(\frac{w}{zq^{i-j+r}}\right) u_{i+r}(z)\, u_{j-r}(w), \end{aligned} \tag{2.71}$$

where $\delta(z) = \sum_{m\in\mathbb{Z}} z^m$. *So this bracket coincides with the one constructed by Frenkel and Reshetikhin in* [6].

Proof of this theorem is straightforward computation.

3. The group of q-pseudodifference symbols of all complex degrees and the associated q-KdV hierarchies.

3.1. **The double extension of the algebra $\Psi\mathbf{D}_q$.** Let us define the operator $\ln D \in \mathrm{End}\left(\mathbb{C}\left(\left(z^{-1}\right)\right)\right)$ by

$$\ln D = \ln q \cdot z\frac{d}{dz}, \tag{3.1}$$

where the branch of $\ln q$ is fixed by

$$-\pi < \arg q < \pi, \quad \ln 1 = 0. \tag{3.2}$$

As above, we suppose that $|q| < 1$. Note that the subspaces $\mathbb{C}z^m$ are the eigenspaces for $\ln D$ with eigenvalues $\lambda_m = m\ln q$, hence the exponential $\exp\ln D$ is well-defined and $\exp\ln D = D$, which justifies our definition. Evidently,

$$[\ln D, f] = \ln q \cdot z\frac{df}{dz}, \quad \forall f \in \mathbb{C}\left(\left(z^{-1}\right)\right), \qquad [\ln D, D] = 0, \tag{3.3}$$

which implies that $[\ln D, \cdot]$ is an (outer) derivation of the associative algebra $\Psi\mathbf{D}_q$.

Proposition 3.1. *The 2-form*

$$\omega\left(X,Y\right)=\left\langle\left[\ln D,X\right],Y\right\rangle,\quad\forall X,Y\in\Psi\mathbf{D}_q,\tag{3.4}$$

is a 2-cocycle on $\Psi\mathbf{D}_q$.

Proof. Note that for any $X\in\Psi\mathbf{D}_q$

$$Tr\left[\ln D,X\right]=0,$$

which, together with (3.3), implies the skew-symmetry of ω. Next we have

$$\left[\ln D,[X,Y]\right]=\left[X,[\ln D,Y]\right]-\left[Y,[\ln D,X]\right],$$

hence

$$\begin{aligned}\omega\left([X,Y],Z\right)&=\left\langle\left[X,[\ln D,Y]\right],Z\right\rangle-\left\langle\left[Y,[\ln D,X]\right],Z\right\rangle\\&=\left\langle\left[\ln D,Y\right],[Z,X]\right\rangle+\left\langle\left[\ln D,X\right],[Y,Z]\right\rangle\\&=-\omega\left([Z,X],Y\right)-\omega\left([Y,Z],X\right),\end{aligned}$$

as desired.

The logarithmic cocycle ω defines a non-trivial central extension $\widehat{\Psi\mathbf{D}_q}=\Psi\mathbf{D}_q\dot{+}\mathbb{C}\cdot\mathbf{c}$ of the Lie algebra $\Psi\mathbf{D}_q$. $\widehat{\Psi\mathbf{D}_q}$ does not admit a non-degenerate invariant inner product. To improve the situation let us consider the "double extension" $\widetilde{\Psi\mathbf{D}_q}$ of the algebra $\Psi\mathbf{D}_q$:

$$\widetilde{\Psi\mathbf{D}_q}=\Psi\mathbf{D}_q\dot{+}\mathbb{C}\cdot\ln D\dot{+}\mathbb{C}\cdot\mathbf{c};\tag{3.5}$$

the bilinear form

$$\left\langle X+\alpha\ln D+\beta\mathbf{c},Y+\gamma\ln D+\delta\mathbf{c}\right\rangle=\left\langle X,Y\right\rangle_{\Psi\mathbf{D}_q}+\alpha\delta+\beta\gamma\tag{3.6}$$

is invariant, non-degenerate and sets into duality the subspaces J_+ and J_-; moreover,

$$J_0^*\approx J_0,\quad\left(\mathbb{C}\cdot\ln D\right)^*\approx\mathbb{C}\cdot\mathbf{c}\tag{3.7}$$

(here $J_\pm, J_0$ are defined by (2.5)–(2.7)).

3.2. **The group of q-pseudodifference symbols of all complex degrees.** For any $\alpha\in\mathbb{C}$ we define the complex power $\hat{h}^\alpha$ of the operator $\hat{h}$ by

$$\left(\hat{h}^\alpha f\right)(z)\equiv{}^{h^\alpha}f\left(z\right)=f\left(q^\alpha z\right),\tag{3.8}$$

where $q^\alpha=\exp\left(\alpha\ln q\right)$ and the branch of $\ln q$ is fixed by (3.2).

A normalized q-pseudodifference symbol of degree α is a formal series of the form

$$L=D^\alpha+\sum_{i=1}^{\infty}a_iD^{\alpha-i},\quad a_i\in\mathbb{C}\left(\left(z^{-1}\right)\right).\tag{3.9}$$

The multiplication law of symbols is uniquely defined by the commutation relation:

$$D^{\alpha} \circ a = {}^{h^{\alpha}}a \circ D^{\alpha}. \tag{3.10}$$

Let $\widehat{G}_{\alpha}$ be the set of normalized q-difference symbols of degree α and $\widehat{G}_{-}$ be the set of symbols of all complex degrees,

$$\widehat{G}_{-} = \bigcup_{\alpha \in \mathbb{C}} \widehat{G}_{\alpha}.$$

$\widehat{G}_{-}$ is a group with respect to the multiplication law (3.10). This group admits the following description. For $\alpha \in \mathbb{C}$ let σ_{α} be the automorphism of $\widehat{G}_{-}$ given by

$$\sigma_{\alpha}(X) = D^{\alpha} X D^{-\alpha}. \tag{3.11}$$

Obviously, σ_{α} preserves the degree of symbols and hence induces an automorphism of the subgroup $\widehat{G}_{0}$.

Lemma 3.2. *The group $\widehat{G}_{-}$ is the semi-direct product of the additive group $\mathbb{C}$ and the group $\widehat{G}_{0}$.*

For $L \in \widehat{G}_{-}$ we shall write $L = \bar{L} D^{\alpha}$, where α is the degree of L and $\bar{L} \in \widehat{G}_{0}$. In an obvious sense, $\widehat{G}_{-}$ may be regarded as an infinite-dimensional Lie group.

Lemma 3.3. *The tangent Lie algebra of the group $\widehat{G}_{-}$ is the algebra $\hat{J}_{-} = J_{-} \dot{+} \mathbb{C} \cdot \ln D$ considered as a Lie subalgebra in $\widetilde{\Psi \mathbf{D}_q}$.*

Proof. We must check that

$$[\ln D, X] = \frac{d}{d\alpha} |_{\alpha=0} D^{\alpha} X D^{-\alpha} \tag{3.12}$$

for all $X \in \Psi \mathbf{D}_q$. Since $[D^{\alpha}, D] = 0$, it is sufficient to prove (3.12) for $X = f, \quad \forall f \in \mathbb{C}((z^{-1}))$. In this case we have

$$\frac{d}{d\alpha} |_{\alpha=0} D^{\alpha} f(z) D^{-\alpha} = \frac{d}{d\alpha} |_{\alpha=0} f(q^{\alpha} z) = \ln q \cdot z \frac{df}{dz} = [\ln D, f],$$

as desired. ■

Let us fix the following models of the tangent and cotangent spaces of $\widehat{G}_{-}$ which will be used over the rest of this section:

$$\begin{aligned} T_L \widehat{G}_{-} &= \left\{ X = \overline{X} D^{\alpha} + \widetilde{X} L \ln D, \quad \overline{X} \in J_{-}, \quad \widetilde{X} \in \mathbb{C} \right\}, \\ T_L^{*} \widehat{G}_{-} &= \left\{ f = D^{-\alpha} \bar{f} + \tilde{f} \mathbf{c} L^{-1}, \quad \bar{f} \in J_{+}, \quad \tilde{f} \in \mathbb{C} \right\}. \end{aligned} \tag{3.13}$$

The pairs $\left(\overline{X}, \widetilde{X}\right)$ and $\left(\bar{f}, \tilde{f}\right)$ will be called *normal coordinates* of X and f, respectively. The canonical pairing between $T_L \widehat{G}_{-}$ and $T_L^{*} \widehat{G}_{-}$ is given by

$$\langle X, f \rangle = \langle \overline{X}, \bar{f} \rangle_{\Psi \mathbf{D}_q} + \widetilde{X} \cdot \tilde{f}. \tag{3.14}$$

Let us denote by ${}^{l}S_L$ and ${}^{r}S_L$ the operators of the left (resp., right) multiplication by L in $\widehat{G}_-$.

Lemma 3.4. *In normal coordinates the tangent map $\left({}^{r}S_L\right)_*$ at the unit element of $\widehat{G}_-$*

$$\left({}^{r}S_L\right)_{*(e)} : T_e\widehat{G}_- \to T_L\widehat{G}_-, \quad X \mapsto Y,$$

is given by:

$$\begin{array}{rcl} \overline{Y} & = & \overline{X}\bar{L} + \widetilde{X}\left[\ln D, \bar{L}\right], \\ \widetilde{Y} & = & \widetilde{X}. \end{array} \tag{3.15}$$

The tangent map $\left({}^{l}S_L\right)_{*(e)}$ is given by

$$\begin{array}{rcl} \overline{Y} & = & \bar{L}\sigma_\alpha\left(\overline{X}\right), \\ \widetilde{Y} & = & \widetilde{X}. \end{array} \tag{3.16}$$

Formulae (3.16) may be formally written as $Y = LX$.

Proposition 3.5.

1) The exponential map $\exp : \hat{J}_- \to \widehat{G}_-$ *is well defined on the whole $\hat{J}_-$.*
2) The restriction of the exponential map to the affine subspace $\hat{J}_\alpha = J_- + \alpha \ln D$ is a bijection between $\hat{J}_\alpha$ and $\widehat{G}_\alpha$ if

$$\frac{\alpha \ln q}{2\pi i} \notin \mathbb{Q}. \tag{3.17}$$

Proof. By definition, $L(t) = \exp t\left(X + \ln D\right), \quad X \in J_-$, is a solution of the following differential equation

$$\frac{dL}{dt} = \left({}^{l}S_L\right)_{*(e)}\left(X + \ln D\right) \tag{3.18}$$

with the initial condition $L(0) = e$. Evidently, $L(t)$ has the form $L(t) = A(t)\,D^t$, where $A(t) \in \widehat{G}_0$. By lemma 3.4 we write (3.18) as

$$\frac{dA}{dt} = A\sigma_t\left(X\right), \quad t \in \mathbb{C}. \tag{3.19}$$

This equation has a unique solution with initial condition $A(0) = e$. Indeed, let $A_i^j(t)$, X_i^j be the coefficients of the expansion of $A(t)$ (respectively, X) in terms of z and D :

$$\begin{array}{rclrcl} A(t) & = & \sum\limits_{i=0}^{\infty} A_i(t)\,D^{-i}, & A_i(t) & = & \sum\limits_{j=-\infty}^{m_i} A_i^j(t)\,z^j,\ j \geq 1,\ A_0 = 1; \\ X & = & \sum\limits_{i=1}^{\infty} X_i D^{-i}, & X_i & = & \sum\limits_{j=-\infty}^{n_i} X_i^j z^j. \end{array}$$

(We set $A_i^j(t) \equiv 0,\ j > m_i, \quad X_i^j \equiv 0,\ j > n_i$, and extend summation up to infinity.) From (3.19) we obtain

$$\frac{dA_1^m}{dt} = q^{tm} X_1^m, \quad A_1^m(0) = 0; \tag{3.20}$$

$$\frac{dA_i^m}{dt} = \sum_{j=o}^{i-1} \sum_{n\in\mathbb{Z}} A_j^n q^{nt} X_{i-j}^{m-n} + q^{mt} X_i^m, \quad A_i^m(0) = 0. \tag{3.21}$$

We shall prove inductively that this system admits a unique solution which is holomorphic in $\mathbb{C}$. Observe first of all that the function $w \longmapsto q^{mw}$ is holomorphic in $\mathbb{C}$, hence the value of the integral $\int_0^t q^{mw}dw$ does not depend on the path of integration and

$$A_1^m(t) = \int_0^t q^{mw} dw \cdot X_1^m \tag{3.22}$$

gives the unique solution of (3.20). Obviously, $A_1^m(t)$ is holomorphic in $\mathbb{C}$.

Assume now that all coefficients $A_1, \ldots, A_{i-1}$ are holomorphic functions. We shall deduce from it that the equation (3.21) for A_i is also solvable in holomorphic functions. Indeed, the sum over n in the r.h.s. of (3.21) has only a finite number of non-zero terms and the functions A_j^n are holomorphic by the inductive hypothesis, hence the r.h.s. of (3.21) is holomorphic and this equation has the unique solution which is given by

$$A_i^m(t) = \int_0^t \left(\sum_{j=o}^{i-1} \sum_{n\in\mathbb{Z}} A_j^n(w)\, q^{nw} X_{i-j}^{m-n} + q^{mw} X_i^m \right) dw. \tag{3.23}$$

Thus, the exponential map is well-defined. To verify the second assertion of the proposition we need to prove that for any $L \in \widehat{G}_\alpha$, $\frac{\alpha \ln q}{2\pi i} \notin \mathbb{Q}$, there is a unique representation of the form

$$L \equiv AD^\alpha = \exp \alpha (X + \ln D), \quad X \in J_-.$$

Consider (3.22), (3.23) as an equation for X. We have

$$A_1^m = \int_0^\alpha q^{mw} dw \cdot X_1^m = \begin{cases} \frac{e^{\alpha m \ln q} - 1}{m \ln q} X_1^m, & m \neq 0, \\ \alpha X_1^0, & m = 0. \end{cases} \tag{3.24}$$

Since$\frac{\alpha \ln q}{2\pi i} \notin \mathbb{Q}$, , $e^{\alpha m \ln q} \neq 1$, and hence (3.24) is solvable for any A_1^m. Let us assume now that $X_1, \ldots, X_{i-1}$ are already determined. In that case equation (3.23) for X_i has a unique solution. Indeed, we have

$$A_i^m - \int_0^\alpha \left(\sum_{j=o}^{i-1} \sum_{n\in\mathbb{Z}} A_j^n(w)\, q^{nw} X_{i-j}^{m-n} \right) dw = \int_0^\alpha q^{mw} dw \cdot X_i^m. \tag{3.25}$$

Functions $A_j^n(w)$ are defined by the formulae (3.23) and may be expressed in terms of $X_1, \dots, X_{i-1}$, hence the l.h.s. of (3.25) is a known number. As above, we see that (3.25) is uniquely solvable provided that $\frac{\alpha \ln q}{2\pi i} \notin \mathbb{Q}$.

Definition 3.1. *We call $\alpha \in \mathbb{C}$ generic if $\frac{\alpha \ln q}{2\pi i} \notin \mathbb{Q}$; we call $L \in \widehat{G}_-$ a generic element if its degree $\alpha \equiv \deg L$ is generic.*

3.3. **The generalized q-deformed Gelfand-Dickey structure on $\widehat{G}_-$ and related q-KdV hierarchies.** In this section we consider Lax equations of the form

$$\frac{dL}{dt} = \left[L_{(+)}^{\frac{m}{\alpha}}, L\right], \quad L \in \widehat{G}_\alpha, \quad \alpha \text{ is generic.} \tag{3.26}$$

We shall see that in some natural class of quadratic Poisson brackets on $\widehat{G}_\alpha$ there exists a unique one which is consistent with the equations (3.26) (i.e., the latter are Hamiltonian with respect to this bracket with the Hamiltonians $H_m = \frac{\alpha}{m} TrL^{\frac{m}{\alpha}}$). For any positive integer α this bracket may be restricted to $\mathbb{M}_\alpha$ and coincides there with the q-deformed Gelfand-Dickey structure considered in part 1.

The brackets referred to are smooth with respect to the parameter α outside the line $\frac{2\pi i}{\ln q}\mathbb{R}$, hence we can glue them up to a smooth Poisson structure on $\widehat{G}'_- = \bigcup_{\alpha \notin \frac{2\pi i}{\ln q}\mathbb{R}} \widehat{G}_\alpha$. This bracket is called the *generalized q-deformed (second) Gelfand-Dickey structure*. It is uniquely defined by the following conditions:

1) $\widehat{G}_\alpha$ are Poisson submanifolds;
2) the restriction of this bracket to $\widehat{G}_\alpha$ coincides with the unique bracket on $\widehat{G}_\alpha$ consistent with equation (3.26).

Let us now turn back to equation (3.26).

Theorem 3.6.

1. *The equation (3.26) is self-consistent, i.e., its r.h.s. is well-defined and belongs to the tangent space $T_L\widehat{G}_\alpha$;*
2. *The flows corresponding to different m 's commute with each other;*
3. *The functionals*

$$H_n = \frac{\alpha}{n} TrL^{\frac{n}{\alpha}}, \quad n \in \mathbb{N}, \tag{3.27}$$

are conservation laws for (3.26).

Proof. $L \in \widehat{G}_\alpha$ and α is generic, hence in agreement with proposition 3.5 there exists an $X \in J_-$ such that $L = \exp \alpha (X + \ln D)$. For any $\beta \in \mathbb{C}$ we define L^β by $L^\beta = \exp \alpha\beta (X + \ln D)$; clearly, $L^\beta \in \widehat{G}_{\alpha\beta}$. In particular, $L^{\frac{m}{\alpha}} \in \widehat{G}_m$, hence it contains only integer powers of D and may be considered as an element of $\Psi\mathbf{D}_q$. So the expressions $L_{(+)}^{\frac{m}{\alpha}}$ and $TrL^{\frac{n}{\alpha}}$ are well-defined. Note that $\left[L^{\frac{m}{\alpha}}, L\right] = 0$,

therefore $\left[L^{\frac{m}{\alpha}}_{(+)}, L\right] = -\left[L^{\frac{m}{\alpha}}_{-}, L\right]$ has the form $\sum_{i=1}^{\infty} a_i D^{\alpha-i}$, i.e., it is a tangent vector from $T_L\widehat{G}_\alpha$. The last two assertions of the theorem may be proved in the same way as in the case of positive integer α. Another proof will be given below in the frameworks of the Hamiltonian formalism.

Remark 3.1. *A similar class of equations has been constructed in* [11]*. The authors start with the algebra ΨDO_q of q–difference symbols of the form*

$$A = \sum_{i=-\infty}^{n(A)} u_i(z) D_q^i, \quad u_i \in \mathbb{C}\left[z, z^{-1}\right], \tag{3.28}$$

with the commutation relation

$$D_q \circ u = \frac{u(qz) - u(z)}{q-1} +^h u \cdot D_q,$$

which corresponds to the definition of D_q as a q-difference analogue of the derivative $\frac{d}{dz}$:

$$(D_q f)(z) = \frac{f(qz) - f(z)}{q-1}.$$

Then they define the residue of a symbol $A \in \Psi DO_q$ by $\widetilde{\mathrm{Res}} A = u_{-1}(z)$ (see (3.28)); the trace is given by

$$\widetilde{\mathrm{Tr}} A = \int \frac{dz}{z} \widetilde{\mathrm{Res}} \left(\frac{A}{(q-1) D_q + 1} \right).$$

Let us extend ΨDO_q by assuming that the coefficients in (3.28) are formal power series in z^{-1}. It is easy to see that after this extension ΨDO_q becomes isomorphic to $\Psi \mathbf{D}_q$ (as an associative algebra) and, moreover, the definitions of the traces in ΨDO_q and $\Psi \mathbf{D}_q$ coincide up to a scalar factor.

Then the authors of [11] *construct a double extension of ΨDO_q by adjoining to it the logarithm $\log D_q$ and the corresponding 2-cocycle, and define the group G_q of q-difference symbols of the form*

$$F = D_q^\alpha + \sum_{k=1}^{\infty} u_k(z) D_q^{\alpha-1}.$$

We did not found an isomorphism between G_q and $\widehat{G}_-$; the question about the relation between the generalized q-KdV equations (3.26) and the ones constructed in [11] *remains open.*

We will now examine the Hamiltonian formalism for equations (3.26). Let us consider the set $\widehat{G}_\alpha$, where α is generic. Just as in part 1, we consider the Poisson

brackets of the form

$$\{\varphi,\psi\} = \left\langle\left\langle \begin{pmatrix} r + aP_0 & bP_0 \\ cP_0 & r + dP_0 \end{pmatrix} \begin{pmatrix} \overline{\nabla\varphi} \\ \overline{\nabla'\varphi} \end{pmatrix}, \begin{pmatrix} \overline{\nabla\psi} \\ \overline{\nabla'\psi} \end{pmatrix} \right\rangle\right\rangle, \tag{3.29}$$

where a, b, c, d are linear operators in J_0 satisfying the skew-symmetry conditions

$$a = -a^*, \quad d = -d^*, \quad c^* = b; \tag{3.30}$$

the r-matrix r is defined as above: $r = \frac{1}{2}(P_+ - P_-)$; $\overline{\nabla\varphi}$, $\overline{\nabla'\varphi}$ denote the normal coordinates of $\nabla\varphi$, $\nabla'\varphi$. Note that for a functional $\varphi \in Fun\left(\widehat{G}_\alpha\right)$ the component $\overline{d\varphi}$ of its linear gradient is defined up to an arbitrary element of $J_{(-)}$, and $\widetilde{d\varphi}$ is arbitrary. Lemma 3.4 implies that

$$\overline{\nabla\varphi} = \overline{L}\,\overline{d\varphi}, \quad \overline{\nabla'\varphi} = \sigma_{-\alpha}\left(\overline{d\varphi\,\overline{L}}\right), \tag{3.31}$$

where $\overline{L} = LD^{-\alpha}$. Hence, $\widetilde{d\varphi}$ gives no contribution to the bracket (3.29).

Theorem 3.7. *There exists a unique Poisson bracket of the form (3.29) on $\widehat{G}_\alpha$ which satisfies the following conditions:*

1) the expression (3.29) is well-defined, i.e., it does not depend on the $J_{(-)}$-components of $\overline{d\varphi}$, $\overline{d\psi}$;

2) the Hamiltonians H_m (see (3.27)) give rise to the Lax equations (3.26).

This bracket is given by

$$\{\varphi,\psi\} \;=\; \left\langle\left\langle \begin{pmatrix} r + \frac{1}{2}\frac{1+\hat{h}^\alpha}{1-\hat{h}^\alpha}P_0' & -\frac{\hat{h}^\alpha}{1-\hat{h}^\alpha}P_0' + \frac{1}{2}P_{00} \\ \frac{1}{1-\hat{h}^\alpha}P_0' + \frac{1}{2}P_{00} & r - \frac{1}{2}\frac{1+\hat{h}^\alpha}{1-\hat{h}^\alpha}P_0' \end{pmatrix} \begin{pmatrix} \overline{\nabla\varphi} \\ \overline{\nabla'\varphi} \end{pmatrix}, \begin{pmatrix} \overline{\nabla\psi} \\ \overline{\nabla'\psi} \end{pmatrix} \right\rangle\right\rangle. \tag{3.32}$$

Proof. We shall seek for a, b, c, d such that the bracket (3.29) satisfies the two conditions of the theorem.

Lemma 3.8.

$$\overline{dH_m} = D^\alpha L^{\frac{m}{\alpha}-1}. \tag{3.33}$$

Lemma 3.8 and (3.31) imply that

$$\overline{\nabla H_m} = \overline{\nabla' H_m} = L^{\frac{m}{\alpha}}. \tag{3.34}$$

Substituting this into (3.29), we obtain the following

Proposition 3.9. *Condition (2) of the theorem is equivalent the following one: for any $L \in \widehat{G}_\alpha$, any $m \in \mathbb{N}$, holds*

$$\left(\left[a + b - \frac{1}{2}\right]\,\left(L^{\frac{m}{\alpha}}\right)_0\right)\cdot L = L\cdot\left(\left[c + d - \frac{1}{2}\right]\,\left(L^{\frac{m}{\alpha}}\right)_0\right). \tag{3.35}$$

Lemma 3.10. *Condition (3.35) implies that*

$$a+b-1/2=(c+d-1/2)=F, \tag{3.36}$$

where F is a linear operator in $\mathbb{C}((z^{-1}))$, $Im F=\mathbb{C}\cdot 1\subset\mathbb{C}((z^{-1}))$.

Proof. Since $|q|<1$, the point 1 is generic and by proposition 3.5 the map $L\mapsto L^{\frac{1}{\alpha}}$ is a bijection between $\widehat{G}_\alpha$ and $\widehat{G}_1$. Therefore, for any $f\in\mathbb{C}((z^{-1}))$, $f\neq 0$, there exists an $L\in\widehat{G}_\alpha$ such that $\left(L^{\frac{1}{\alpha}}\right)_0=f$; moreover, relation (3.24) implies that in the expansion

$$L=D^\alpha+u_1D^{\alpha-1}+\cdots \tag{3.37}$$

the coefficient u_1 is nonzero. Put $\tilde F=(a+b-1/2)f$ and $\tilde G=(c+d-1/2)f$. We have $\tilde F L=L\tilde G$, which implies $\tilde F=\hat h^\alpha\tilde G$ and $u_1\tilde F=u_1\hat h^{\alpha-1}\tilde G$. But $u_1\neq 0$, hence $\tilde F=\hat h^\alpha\tilde G=\hat h^{\alpha-1}\tilde G$, i.e. $\tilde F\in\mathbb{C}$, $\tilde G\in\mathbb{C}$, which, together with (3.35), implies that $\tilde F=\tilde G$. ■

Using the skew-symmetry conditions (3.30) we get

$$b=\frac{1}{2}-a+F,\quad c=\frac{1}{2}+a+F^*,\quad d=-a+F-F^*. \tag{3.38}$$

Just as in the proof of theorem 2.7 we can verify that F does not contribute to the Poisson bracket. The invariance of the inner product now implies that

$$\{\varphi,\psi\}=\left\langle\!\!\left\langle\begin{pmatrix} P_++\left(\frac{1}{2}+a\right)P_0 & \left(\frac{1}{2}-a\right)P_0\\ \left(\frac{1}{2}+a\right)P_0 & P_++\left(\frac{1}{2}-a\right)P_0\end{pmatrix}\begin{pmatrix}\overline{\nabla\varphi}\\ \overline{\nabla'\varphi}\end{pmatrix},\begin{pmatrix}\overline{\nabla\psi}\\ \overline{\nabla'\psi}\end{pmatrix}\right\rangle\!\!\right\rangle. \tag{3.39}$$

Hence $\{\varphi,\psi\}$ does not depend on J_--components of $\overline{d\varphi}$, $\overline{d\psi}$. The requirement that it is also independent on J_0-components of the gradients fixes the choice of a.

Lemma 3.11. *The bracket (3.39) does not depend on J_0-components of $\overline{d\varphi}$, $\overline{d\psi}$ if and only if*

$$\left[\left(\frac{1}{2}+a\right)+\left(\frac{1}{2}-a\right)\hat h^{-\alpha}\right]f\in\mathbb{C}.\quad \textit{for } \forall f\in\mathbb{C}((z^{-1})).$$

Proof. Let $\overline{d\varphi}'$ be another representative of $\overline{d\varphi}$, $\overline{d\varphi}'=\overline{d\varphi}+f$, $\quad f\in J_0$, and $\{\varphi,\psi\}'$ be the value of the Poisson bracket corresponding to $\overline{d\varphi}'$. We must prove that

$$\Delta=\{\varphi,\psi\}'-\{\varphi,\psi\}=0.$$

We have

$$\Delta=\left\langle\!\!\left\langle\begin{pmatrix}\frac{1}{2}+a & \frac{1}{2}-a\\ \frac{1}{2}+a & \frac{1}{2}-a\end{pmatrix}\begin{pmatrix}\left(\bar L f\right)_0\\ \sigma_{-\alpha}\left(f\,\bar L\right)_0\end{pmatrix},\begin{pmatrix}\overline{\nabla\psi}\\ \overline{\nabla'\psi}\end{pmatrix}\right\rangle\!\!\right\rangle. \tag{3.40}$$

Since $f \in J_0$, we have $\left(\bar{L} f\right)_0 = f$ and $\sigma_{-\alpha}\left(f\,\bar{L}\right)_0 = \hat{h}^{-\alpha} f$, hence

$$\Delta = \left\langle \left[\left(\frac{1}{2}+a\right)+\left(\frac{1}{2}-a\right)\hat{h}^{-\alpha}\right] f\,,\left(\overline{\nabla\psi}\right)_0 - \left(\overline{\nabla'\psi}\right)_0 \right\rangle .$$

Lemma 3.12. *For any $g \in \mathbb{C}((z^{-1}))$ such that $\int \frac{dz}{z} g(z) = 0$ there exists a functional $\psi_g \in Fun\left(\hat{G}_\alpha\right)$ such that for some $L \in \hat{G}_\alpha$*

$$\left(\overline{\nabla\psi_g}\,(L)\right)_0 - \left(\overline{\nabla'\psi_g}\,(L)\right)_0 = g.$$

Proof. Note that either $\alpha - 1$ or $\alpha - 2$ are generic. If $\alpha - 1$ is generic, we may suppose that

$$\psi_g = Tr\bar{L}Da_g, \quad a_g \in \mathbb{C}((z^{-1})).$$

It is easy to see that in this case

$$\left(\overline{\nabla\psi_g}\,(L)\right)_0 - \left(\overline{\nabla'\psi_g}\,(L)\right)_0 = \left(1-\hat{h}^{1-\alpha}\right)(u_1 a_g)\,, \tag{3.41}$$

where u_1 is the coefficient in the expansion of $\bar{L}$ in powers of D^{-1} :

$$\bar{L} = 1 + u_1 D^{-1} + \cdots$$

The condition that $\alpha - 1$ is generic implies that the equation $\left(1-\hat{h}^{1-\alpha}\right)(u_1 a_g) = g$ is solvable for any $g \in \mathbb{C}((z^{-1}))$ such that $\int \frac{dz}{z} g(z) = 0$.

If $\alpha - 2$ is generic we can find ψ_g in the form $\psi_g = Tr\bar{L}D^2 a_g$. ∎

Δ must vanish for any $f \in \mathbb{C}((z^{-1}))$ and any $\psi \in Fun\left(\hat{G}_\alpha\right)$, therefore from lemma 3.12 it follows that

$$\left[\left(\frac{1}{2}+a\right)+\left(\frac{1}{2}-a\right)\hat{h}^{-\alpha}\right] f \in \mathbb{C} \quad \text{for } \forall f \in \mathbb{C}((z^{-1})).$$

End of the proof of this theorem is just like the one of theorem 2.7. ∎

Remark 3.2. *As in part 1 we can linearize the bracket (3.32) at $L = D^\alpha$ and construct the family of compatible Poisson structures.*

Now we can prove assertions 2 and 3 of theorem 3.6; they immediately result from the following

Proposition 3.13. *Functionals $H_n = \frac{\alpha}{n} Tr L^{\frac{n}{\alpha}}$, $n \in \mathbb{N}$, are in involution with respect to the bracket (3.32).*

Proposition 3.14. *The submanifolds $\hat{G}_{\alpha,n} \subset \hat{G}_\alpha$ of the symbols of the form*

$$L = \left(1 + \sum_{i=1}^{n} u_i D^{-i}\right) D^\alpha \tag{3.42}$$

are Poisson submanifolds for bracket (3.32).

Proof. It is sufficient to check that the bracket of any functional of the form

$$\varphi_{f,l} = \int \frac{dz}{z} u_l\, f, \quad f \in \mathbb{C}((z^{-1})), \quad l > n, \tag{3.43}$$

with any functional ψ vanishes on $\hat{G}_{\alpha,n}$. Clearly, $\overline{d\varphi_{f,l}} = D^l f$, hence $\overline{\nabla\varphi_{f,l}}(L)$, $\overline{\nabla'\psi_{f,l}}(L) \in J_+$ for all $L \in \hat{G}_{\alpha,n}$ and we have

$$\{\varphi_{f,l}\,,\psi\} = \frac{1}{2}\left(\left\langle\overline{\nabla\varphi_{f,l}}\,,\overline{\nabla\psi}\right\rangle - \left\langle\overline{\nabla'\varphi_{f,l}}\,,\overline{\nabla'\psi}\right\rangle\right) = 0,$$

as desired.

Proposition 3.15. *The coefficient $u_n(z)$ is a Casimir function on $\hat{G}_{n,n}$.*

Proof. Define $\varphi_{f,n}$ by (3.43). We shall prove that $\{\varphi_{f,n}\,,\psi\}$ vanishes on $\hat{G}_{n,n}$. We may write the bracket (3.32) in the form

$$\begin{aligned}&\{\varphi_{f,n}\,,\psi\} = \\ &= \left\langle\left\langle \begin{pmatrix} -P_- + \left(a-\frac{1}{2}\right)P_0 & \left(\frac{1}{2}-a\right)P_0 \\ \left(a+\frac{1}{2}\right)P_0 & -P_- - \left(a+\frac{1}{2}\right)P_0 \end{pmatrix} \begin{pmatrix} \overline{\nabla\varphi} \\ \overline{\nabla'\varphi} \end{pmatrix}, \begin{pmatrix} \overline{\nabla\psi} \\ \overline{\nabla'\psi} \end{pmatrix} \right\rangle\right\rangle.\end{aligned} \tag{3.44}$$

Clearly, $\overline{\nabla\varphi_{f,n}}\ ,\ \overline{\nabla'\varphi_{f,n}} \in J_{(+)}$, hence only the J_0-components of the gradients give contribution to the bracket. We have

$$\left(\overline{\nabla'\varphi_{f,n}}\right)_0 = \sigma_{-n}\left(\left(\overline{d\varphi}\,\bar{L}\right)_0\right) = \sigma_{-n}\left(D^n f\, u_n D^{-n}\right) = f\,u_n = \left(\overline{\nabla\varphi_{f,n}}\right)_0.$$

Substituting this in (3.44) we obtain $\{\varphi_{f,n}\,,\psi\} = 0$, as desired.

Note that for integer n the submanifold $\hat{G}_{n,n}$ may be canonically identified with the subspace $\mathbb{M}_n \subset \Psi\mathbf{D}_q$ considered in part 1 of this paper. With this identification the restriction of the bracket (3.32) on $\hat{G}_{n,n}$ coincides with the bracket (2.18). Indeed, the r-matrices are the same and we need only to check that the definitions of the gradients are consistent with each other. We have

$$\begin{aligned}\overline{\nabla\varphi} &= \bar{L}\overline{d\varphi} = \bar{L}D^n \cdot D^{-n}\overline{d\varphi} = Ld\varphi, \\ \overline{\nabla'\varphi} &= \sigma_{-n}\left(\overline{d\varphi}\,\bar{L}\right) = D^{-n}\overline{d\varphi}\cdot\bar{L}D^n = d\varphi L,\end{aligned}$$

as desired.

4. Jacobi identity for the generalized q-deformed Gelfand-Dickey structures.

In this section we discuss the Jacobi identity for generalized q-deformed Gelfand-Dickey brackets (2.18), (3.32). We start with the following general pattern, due to Gelfand and Dorfman [9]. Let $\mathcal{A} = \Omega_0$ be an associative commutative algebra, $\mathfrak{a} = Der\mathcal{A}$ the Lie algebra of its derivations; we regard $\mathcal{A}$ as a $\mathfrak{a}$-module and

define the Chevalley complex associated with $\mathcal{A}$ in the standard way,

$$\Omega_0 \xrightarrow{d} \Omega_1 \xrightarrow{d} \Omega_2 \xrightarrow{d} ..., \Omega_p = \mathcal{A} \otimes \bigwedge^p \mathfrak{a}^*,$$
$$d\alpha(X_1, ..., X_{p+1}) = \sum_i (-1)^i X_i \alpha(X_1, ..., \hat{X}_i, ..., X_{p+1}) +$$
$$\sum_{i<j} (-1)^{i+j} \alpha(X_1, ..., \hat{X}_i, ..., \hat{X}_j, ... X_{p+1}).$$

For $X \in \mathfrak{a}$ let $i_X : \Omega_p \to \Omega_{p-1}$be the inner derivative,

$$i_X \alpha \left(X_1, X_2, ..., X_{p-1}\right) = \alpha \left(X, X_1, X_2, ..., X_{p-1}\right);$$

for $p = 1$ the coupling $\langle X, \alpha \rangle = i_X \alpha$ is a natural bilinear pairing between $\mathfrak{a}$ and Ω_0. Let $L_X = d \cdot i_X + i_X \cdot d$ be the Lie derivative. By definition, a Poisson operator is a linear operator $H \in Hom(\ \Omega_1, \mathfrak{a})$; the Schouten bracket of two Poisson operators H, K is a trilinear mapping $\Omega_0 \times \Omega_0 \times \Omega_0 \to \mathfrak{a}$ defined by

$$[[H, K]] \left(\alpha_1, \alpha_2, \alpha_3\right) = \langle H L_{K\alpha_1} \alpha_2, \alpha_3 \rangle + \langle K L_{H\alpha_1} \alpha_2, \alpha_3 \rangle + c.p.$$

(as usual, *c.p.* stands for cyclic permutation). The Poisson bracket associated with H is given by

$$\{\varphi, \psi\} = \langle H d\varphi, d\psi \rangle; \tag{4.1}$$

this bracket is skew and satisfies the Jacobi identity if and only if H is skew-symmetric and its Schouten bracket with itself is zero.

Let $\mathfrak{J}$ be an associative algebra with a non-degenerated invariant inner product. We choose as $\mathcal{A} = \Omega_0$ the algebra of smooth functionals on $\mathfrak{J}$. (Recall that by definition a functional $\varphi \in \mathcal{A}$ is smooth if for each $L \in \mathfrak{J}$ there exists an element $X \in \mathfrak{J}$ (called its linear gradient) such that

$$\langle d\varphi(L), X \rangle = \left(\frac{d}{dt}\right)_{t=0} \varphi\left(L + tX\right).$$

The left and right gradients ∇, ∇' are given by $\nabla\varphi(L) = Ld\varphi(L), \quad \nabla'\varphi(L) = d\varphi(L)L$. For a functional φ we write $D\varphi = \begin{pmatrix} \nabla\varphi \\ \nabla'\varphi \end{pmatrix} \in \mathfrak{J} \bigoplus \mathfrak{J}$.) Let us define the invariant inner product in $\mathfrak{J} \bigoplus \mathfrak{J}$ by

$$\left\langle\left\langle \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \right\rangle\right\rangle = \langle X_1, Y_1 \rangle - \langle X_2, Y_2 \rangle. \tag{4.2}$$

We are interested in the class of Poisson brackets $\mathfrak{J}$ of the form

$$\{\varphi, \psi\} = \left\langle\left\langle R \begin{pmatrix} \nabla\varphi \\ \nabla'\varphi \end{pmatrix}, \begin{pmatrix} \nabla\psi \\ \nabla'\psi \end{pmatrix} \right\rangle\right\rangle \tag{4.3}$$

where $R \in End \mathfrak{J} \bigoplus \mathfrak{J}\ , \quad R = -R^*$.

Lemma 4.1. *The obstruction term in the Jacobi identity for the bracket (4.3) is given by*

$$\Delta \equiv \{\{\varphi, \psi\}, \chi\} + \text{c.p.} = \langle\langle [RD\varphi, RD\psi], D\chi \rangle\rangle + \text{c.p.} \tag{4.4}$$

Theorem 4.2. *Let $R \in End\left(\mathfrak{J} \bigoplus \mathfrak{J}\right)$ be a skew-symmetric classical r-matrix satisfying the modified classical Yang-Baxter equation (mCYBE) in $\mathfrak{J} \bigoplus \mathfrak{J}$:*

$$[R\mathbb{X}, R\mathbb{Y}] - R\left([R\mathbb{X}, \mathbb{Y}] + [\mathbb{X}, R\mathbb{Y}]\right) = -\frac{1}{4}\left[\mathbb{X}, \mathbb{Y}\right], \quad \forall \mathbb{X}, \mathbb{Y} \in \mathfrak{J} \bigoplus \mathfrak{J}.$$

Then the bracket (4.4) satisfies the Jacobi identity.

Proof.

Lemma 4.3. *If R satisfies mCYBE, then*

$$\langle\langle [R\mathbb{X}, R\mathbb{Y}], \mathbb{Z}\rangle\rangle + \text{c.p.} = -\frac{1}{4}\langle\langle [\mathbb{X}, \mathbb{Y}], \mathbb{Z}\rangle\rangle . \tag{4.5}$$

Let φ, ψ, χ be some linear functionals on $\mathfrak{J}$. Put

$$\mathbb{X} = D\varphi, \quad \mathbb{Y} = D\psi, \quad \mathbb{Z} = D\chi. \tag{4.6}$$

By lemmas 4.1, 4.3 we have

$$\begin{aligned} -4\Delta &= \langle\langle [\mathbb{X}, \mathbb{Y}], \mathbb{Z}\rangle\rangle = \langle [\nabla\varphi, \nabla\psi], \nabla\chi\rangle - \langle [\nabla'\varphi, \nabla'\psi], \nabla'\chi\rangle \\ &= \langle [\nabla\varphi, \nabla\psi], \nabla\chi\rangle - Trd\varphi L d\psi L d\chi L + Trd\psi L d\varphi L d\chi L \\ &= \langle [\nabla\varphi, \nabla\psi], \nabla\chi\rangle - TrLd\varphi L d\psi L d\chi + TrLd\psi L d\varphi L d\chi = 0 \end{aligned}$$

as desired.

Theorem 4.4. *Let $\mathfrak{J}$ be a Lie algebra which is (as a linear space) the direct sum of the three subalgebras*

$$\mathfrak{J} = \mathfrak{J}_+ \dot{+} \mathfrak{J}_0 \dot{+} \mathfrak{J}_-, \tag{4.7}$$

where $\mathfrak{J}_0$ is abelian and

$$[\mathfrak{J}_\pm, \mathfrak{J}_0] \subset \mathfrak{J}_\pm. \tag{4.8}$$

Let $P_\pm$, P_0 be the projection operators onto $\mathfrak{J}_\pm, \mathfrak{J}_0$ parallel to the complement and a, b, c, d arbitrary linear operators in $\mathfrak{J}_0$. Put $r = \frac{1}{2}\left(P_+ - P_-\right)$. Then the r-matrix

$$R = \begin{pmatrix} r + aP_0 & bP_0 \\ cP_0 & r + dP_0 \end{pmatrix} \in End\left(\mathfrak{J} \bigoplus \mathfrak{J}\right) \tag{4.9}$$

satisfies mCYBE.

Let us now turn back to the algebra $\Psi\mathbf{D}_q$. From theorems 4.2, 4.4 it follows immediately that the q-deformed Gelfand-Dickey bracket (2.18) satisfies the Jacobi identity.

Proposition 4.5. *The bracket (3.32) on $\hat{G}_\alpha$, α is generic, satisfies the Jacobi identity.*

Proof. Consider the following bracket on $\Psi\mathbf{D}_q$:

$$\{\varphi, \psi\} = \left\langle\left\langle \begin{pmatrix} P_+ + \left(\frac{1}{2} + a\right)P_0 & \left(\frac{1}{2} - a\right)\hat{h}^{-\alpha}P_0 \\ \hat{h}^{\alpha}\left(\frac{1}{2} + a\right)P_0 & P_+ + \left(\frac{1}{2} - a\right)P_0 \end{pmatrix}\begin{pmatrix} \nabla\varphi \\ \nabla'\varphi \end{pmatrix}, \begin{pmatrix} \nabla\psi \\ \nabla'\psi \end{pmatrix}\right\rangle\right\rangle, \tag{4.10}$$

where, as above,

$$a = \frac{1}{2}\frac{1+\hat{h}^{\alpha}}{1-\hat{h}^{\alpha}}P_0'. \tag{4.11}$$

Theorems 4.4, 4.2 imply the Jacobi identity for (4.10).

Lemma 4.6. *$\hat{G}_0$ considered as an affine subspace in $\Psi\mathbf{D}_q$ is a Poisson subspace for (4.10).*

Proof. Obviously, $J_{(-)}$ is a Poisson subspace. Any element $A \in J_{(-)}$ has the form

$$A = \sum_{i=0}^{\infty} u_i D^{-i}, \quad u_i \in \mathbb{C}\left(\left(z^{-1}\right)\right). \tag{4.12}$$

It is sufficient to prove that functionals of the form

$$\varphi_f = TrAf, \quad f \in \mathbb{C}\left(\left(z^{-1}\right)\right), \tag{4.13}$$

are central on $\hat{G}_0$, i.e., their Poisson brackets with any other functional vanish identically on $\hat{G}_0$. Clearly, $d\varphi_f = f \in J_0$, hence $\nabla\varphi_f, \nabla'\varphi_f \in J_{(-)}$. So only the J_0-components of $\nabla\varphi_f, \nabla'\varphi_f$ contribute to the bracket (4.10). Taking into account that $u_0 = 1$ on $\hat{G}_0$, we get

$$\left(\nabla\varphi_f\right)_0 = \left(\nabla'\varphi_f\right)_0 = f. \tag{4.14}$$

Then

$$\{\varphi_f, \psi\} = \left\langle \left[\left(a+\tfrac{1}{2}\right) + \left(\tfrac{1}{2}-a\right)\hat{h}^{-\alpha}\right] f, \left(\nabla\psi\right)_0 - \hat{h}^{\alpha}\left(\nabla'\psi\right)_0 \right\rangle. \tag{4.15}$$

But $\left(a+\frac{1}{2}\right) + \left(\frac{1}{2}-a\right)\hat{h}^{-\alpha} = P_{00}$, hence

$$\{\varphi_f, \psi\} = Trf \cdot \left(Tr\nabla\psi - Tr\hat{h}^{\alpha}\left(\nabla'\psi\right)_0\right) = 0$$

due to the invariance of the inner product.

Consider the map $i : \hat{G}_\alpha \to \hat{G}_0$ defined by $i\left(L\right) = \bar{L}$. Obviously, i is a bijection. It is easy to verify that the pullback of the Poisson bracket (4.10) coincides with the bracket (3.32), so the latter satisfies the Jacobi identity.

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